"Convolution of an Fractional integral equation with the I-function as its kernel"

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Abstract

A new class of convolution integral equations whose kernels involve an H-function of several variables, which is defined by a multiple contour integral of the Mellin-Barnes type, is solved. It is also indicated how the main theorem can be specialized to derive a number of (known or new) results on convolution integral equations involving simpler special functions of interest in problems of applied mathematics and mathematical physics.

The object of the present paper is to generalize the solution of the Convolution Integral Equation involving the Fox'S H-function kernel to the case of Generalized H-function kernel. There are some novel functions that form special cases of the generalized H-function but not Fox H-function. In the present paper, we shall give the solution of convolution equations involving two of such novel functions as kernels.

The object of this paper is to solve an integral equation of convolution form having H-function of two variable as it's kernel. Some known results are obtained as special cases

Keywords - Integral equation, convolution, Generalized type geometric function of Multivariables

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1. Definition and Introduction

(i) The Laplace Transform if

$$F(P) = L[f(t);p] = \int_0^\infty e^{-pt} f(t)dt \ , \quad \mathrm{Re}(p) > 0 \qquad ...(1.1)$$

Then F(p) is called the Laplace transform of f(t) with parameter p and is represented by F(p) = f (t) Erdelyi [(3) pp.129-131]

$$L[f(t);p] = F(P) \text{ then } L[e^{-at}f(t)] = F(p+a) \qquad ...(1.2)$$

And if $f(0) = f^{'}(0) = f^{''}(0) = ... = f^{m-1}(0) = 0 \quad , \ f^{n}(t)$

Is continuous and differential, then

$$L[f^{n}(t);p] = P^{n}F(p) \qquad ....(1.3)$$

(ii) If $L\,[\,f_1(t)] \ = F_1(p)$

then $L\,[\,f_2(t)] \ = F_2(p)$

Then convolution theorem for Laplace transform is

$$L\{\int_0^1 f_1(t)\, f_2(t-u)du\} = L\{f_1(t)\}L\{f_2(t)\} = F_1(p).F_2(p) \quad \text{...(1.4)}$$

(iii) The H-Function Defined by Saxena and kumbhat [1] is an extension of Fox's H-Function on specializing the parameters , H-Function can be reduced to almost all the known special function as well as unknown

The Fox's H-Function of one variable is defined and represented in this Paper as follows [see Srivastava et al [2] ,pp 11-13]

$$H[x] = H_{P,Q}^{M,N}\left[x \,/\, \begin{matrix}(a_j,\alpha_j)_{1,P} \\ (b_j,\beta_j)_{1,Q}\end{matrix}\right] = \frac{1}{2\pi\omega}\int_{\theta=N-1} \theta(\xi)x^{\xi}d\xi \quad \textbf{....(1.5)}$$

...(1.6)

$$\theta(\xi) = \frac{\prod_{i=1}^{n}\Gamma b_j - \beta_j\xi \prod_{j=1}^{N}\Gamma 1 - a_j - \alpha_j\xi}{\prod_{i=M=1}^{Q}\Gamma 1 - b_j + \beta_j\xi \prod_{j=N+1}^{P}\Gamma a_j - \alpha_j\xi}$$

For condition of the H-Function of one variable (1.5) and on the contour L we refer to srivastava et al [2]

(V) The H-Function of two variable occurring in this paper is defined and represented as follows [see Srivastava et al [2] ,pp 83-85]

$$H[x,y] = H^{\,0,n_1,m_2,n_2,m_3,n_3}_{\,p_1,q_1,p_2,q_2,p_3,q_3}\left[x_y \,/\, \begin{matrix}(a_j,\alpha_j,A_j)_{1,p_1} & (c_j,z_j)_{1,p_2} & (e_j,E_j)_{1,p_3} \\ (b_j,\beta_j,B_j)_{1,q_1} & (d_j,\delta_j)_{1,q_2} & (f_j,F_j)_{1,q3}\end{matrix}\right]$$

$$= -\frac{1}{4\pi^2}\int_{L_1}\int_{L_2}\phi_1(\xi,\eta)\psi_2(\xi)\,\psi_3(\eta)x(\xi)\,y(\eta)\,d\xi\, d\eta \quad \text{...(1.7)}$$

Where

$$\phi_1(\xi,\eta) = \prod_{j=1}^{n_1}\Gamma(1-a_j+\alpha_j\xi+A_j\eta)$$

$$\times\left[\prod_{j=n_1+1}^{p_1}\Gamma a_j - \alpha_j\xi - A_j\eta \prod_{j=1}^{q_1}\Gamma 1 - b_j - \beta_j\xi + B_j\eta\right] \quad \text{...(1.8)}$$

Where the $\psi_2(\xi)$ and $\psi_3(\eta)$ are defined as (1.6) and for conditions of existence etc. of the $H(x,y)$ we refer to srivastava et al [2]

(vi)Let $\alpha, \beta \text{ and } \eta$ be complex numbers,and let x $\in R_+$= (0, ∞) Follwing Saigo [7} Fractional integral (Re(α)>0 and derivative Re(α) < 0 of first kind of a function f(x) on R_+ are defined respectively in the forms:

$$I_{0,\ x}^{\alpha,\beta,\eta}(f) = \frac{x^{-\alpha-\beta}}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, 2F1\left(\alpha+\beta\,,\ -\eta\ ;\ \alpha\ ;\ 1-\frac{t}{x}\right) f(t)dt\,; \quad ...(1.9) \quad Re(\alpha)>0$$

$$\frac{d^n}{dt^n} I_{0,\ x}^{\alpha+n\ \beta-n\ ,\eta-n}(f)\ ,\ 0< Re(\alpha)+n<1 \quad (n=1, 2, 3, \ldots\ldots\ldots),$$

Where 2F1(a, b; c; z) is Gauss's hypergeometric function

Fractional integral Re(α)>0 and derivative Re(α) < 0 of second kind a function f(x) on R_+ are given by:

$$J_{x,\infty}^{\alpha,\beta,\eta} = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1}\, t^{\alpha-\beta}\, 2F_1\left(\alpha+\beta\,,\ -\eta\ ;\ \alpha\ ;\ 1-\frac{t}{x}\right) f(t)dt\,; \quad ....(1.10) \quad Re(\alpha)>0$$

$$= (-1)^n \frac{d^n}{dt^n} j_{x,\ \infty}^{\alpha+n\ \beta-n\ ,\eta}(f)\ , \quad 0< Re(\alpha)+n<1 \quad (n=1, 2, 3, \ldots),$$

Let α, β,η and λ be complex numbers. Then there hold the following formulae. The R.H.S. has a definite meaning

$$I_{0,\ x}^{\alpha,\beta,\eta}\ t^{\lambda} = \frac{\Gamma(1+\lambda)\Gamma(1+\lambda-\beta+\eta)}{\Gamma(1+\lambda-\beta)\Gamma(1+\lambda+\alpha+\eta)} x^{1-\beta} \quad .....(1.11)$$

provided that Re(α)> max [0, Re(β-η)] — 1, and

$$j_{0,\ x}^{\alpha,\beta,\eta}\ t^{\lambda} = \frac{\Gamma(\beta-\lambda)\Gamma(\eta-\lambda)}{\Gamma(-\lambda)\Gamma(\alpha+\beta+\eta-\lambda)} x^{\lambda-\beta} \quad .....(1.12)$$

Re (α)>0 and Re(λ) < min [Re (β) , Re (η)], Re(α)< 0, 0< Re(α) + n <1 , and

Re(λ)< min [Re(β)-n , Re(η)] , Where n is a positive integer.

The integral (1.10) is a special case of (1.9) where $\alpha^2 + \beta^2 = \lambda^{-1}$ which was already pointed out by Wille [8] moreover more general results than (1.10) can be found out in the literature ([9], [10])

(vii) The I-Function which was recently introduced by sexena [11] is an extension of Fox'S H-function. On Specializing the parameters,I-Function can bereduced to almost all the known special function as wellas unknowns. The I-Function of one variable is further studied by so many researchers notably as vaishya, jain, and verma [15], Sharma and shrivastava [13] , Sharma and Tiwari [14] ,Nair [12] with certain properties series summation, integration etc.

Saxena [11] represent and define the I-Function of one variable as follows

$$I\,[Z] = I^{m,n}_{p_i;\ q_i\ ;\ r}\left[Z \left| \begin{matrix} (a_j, \alpha_j)_{1,n} \dots \dots (a_{ji}, \alpha_{ji})_{n+1,\ p_i} \\ \cdot \\ \cdot \\ \cdot \\ (b_j,\ \ \beta_j)_{1,m} \dots \dots (b_{ji}, \beta_{ji})_{m+1,\ q_i} \end{matrix} \right. \right] \qquad \text{......(1.13)}$$

$$= \frac{1}{2\pi i} \int_L t(s)\ z^s\ ds.$$

Where

$$t(s) = \frac{\prod_{J=1}^{m} \left[b_j - \beta_j s \quad \prod_{j+1}^{n} \left[1 - a_j, + \alpha_j s \right. \right.}{\sum_{J=1} \left[\prod_{J=m+1}^{q_i} \left[1 - bji + \beta jis \quad \prod_{J=n+1}^{p_i} \left[a_{ji} - \alpha_{ji} s \right] \right. \right.} \qquad \text{......(1.14)}$$

The following definition and results will be required in this paper

2. Main Result

Result 1

$$I^{\alpha,\beta,\eta}_{0,x} \left\{ t^{\lambda}\ L\left\{ t^{\alpha}\ I^{m,n}_{pi,qi,r} \left[at^{\lambda} \left| \begin{matrix} (aj, \alpha j) \dots \dots (aji, \alpha ji)_{pi} \\ \cdot \\ \cdot \\ \cdot \\ (bj, \beta j) \dots \dots (bji, \beta ji)_{qi} \end{matrix} \right. \right] p \right. \right\}$$

$$= \frac{x^{\lambda+\alpha-\beta}}{p} \quad I^{m\ \ n+2}_{P;+2\ \ qi+2,r}$$

$$\left[ax^{\lambda} \left| \begin{matrix} (-\lambda - \alpha, \lambda)\ \ (-\lambda - \alpha - \eta + \beta, \lambda) :\ (aj, \alpha j) \dots \dots (aji, \alpha ji)_{pi} \\ \cdot \\ \cdot \\ \cdot \\ (-\lambda - \alpha + \beta, \lambda)(-\lambda - 2\alpha - \eta, \lambda)\ (bj, \beta j) \dots \dots (bji, \beta ji)_{qi} \end{matrix} \right. \right]$$

Provided Re(β) >0 1≥λ>a and Re(1+∝)>0

Result II

$$I_{0,x}^{\alpha,\beta,\eta}\left\{t^{\lambda+vs}\ \ J_{0,\infty}^{\alpha,\beta,\eta}\ \{t^{\rho+us}\ L\ \{t^{h}\ \ I_{p;\ q;,r}^{m,n}\left[zx^{\lambda}\left|\begin{matrix}(aj,\alpha j)\ \ldots\ldots(aji,\alpha ji)_{pi}\\ \cdot\\ \cdot\\ \cdot\\ (bj,\beta j)\ \ldots\ldots(bji,\beta ji)_{qi}\end{matrix}\right.\right]\right\}$$

$$=\frac{x^{\rho-\lambda-2\beta}}{P^{1+n}}\ I_{P;5\ \ q;+4;r}^{m\ \ n+5}\ [zx^{-u-v}\ P^{-k}$$

$$\left|\begin{matrix}(-\lambda,k)\ (1-\beta+\rho,u)(1-\eta+\rho,u)\ (-\lambda,v)\ (aj,\alpha j)\ \ldots\ldots(aji,\alpha ji)_{pi}\\ \cdot\\ \cdot\\ \cdot\\ (1+\rho,u)(1-\alpha-\beta-\eta+\rho,u)\ (-\lambda+\beta,v)\ (-\lambda-\alpha-\eta,v)\ (bj,\beta j)\ \ldots\ldots(bji,\beta ji)_{qi}\end{matrix}\right]$$

Provided Re (α) >0 Re (β) >0 λ, μ >0

$$\mathrm{Re}\left(\alpha+\lambda\tfrac{b_j}{\beta_j}\right)>0\ \ \mathrm{Re}\left(\beta+\mu\tfrac{d_j}{\delta_j}\right)>0\qquad j=1,2\ldots.m\quad k=1,2\ldots.g$$

$$|\arg z_1|<\tfrac{1}{2}\pi\,\Delta_1\qquad |\arg z_2|<\tfrac{1}{2}\pi\,\Delta_2\quad \Delta_1\ \Delta_2>0$$

$$\Delta_1=\sum_{1}^{M_1}b_j-\sum_{M_1+1}^{Q_1}b_j+\sum_{1}^{N_1}a_j-\sum_{N_1+1}^{P_1}a_j\qquad \Delta_2=\sum_{1}^{M_2}d_j-\sum_{M_2+1}^{Q_2}d_j+\sum_{1}^{N_2}c_j-\sum_{N_2+1}^{P_2}c_j$$

Result III

$$I_{0,x}^{\alpha,\beta,\eta}\{t^{\lambda}\ L\{e^{-nx}\ t^{h}\ I_{p;\ q;\ r}^{m.n}\left[zt^{k}\left|\begin{matrix}(a_j\ \alpha_j)\ldots\ldots(a_{ji}\ \alpha_{ji})_{p_i}\\ \\ (b_{ji},\beta_{ji})_{qi}\end{matrix}\right.\right]p$$

$$=\ (p+n)^{-1-\lambda-h+\beta}\ I_{p;+3,;\ +2;r}^{m,n+3}$$

$$\left[z(p+n)^{-k}\left|\begin{matrix}(\lambda-h,k)(\lambda-h+\beta-n,k)(\lambda-h+\beta,k)(a_j\ \alpha_j 0\ \ldots\ldots(a_{ji}\ \ \alpha_{ji})p_i\\ \cdot\\ \cdot\\ \cdot\\ (b_j\ \ \beta_j)\ \ldots\ldots(b_{ji},\beta_{ji})_{qi}\ (\lambda-h+\beta,k)(\lambda-h-\alpha-\eta,k)\end{matrix}\right.\right]$$

Provided Re(p) >0 1≥λ>a and Re(1+∝)>0

Proof I First Taking by mellin barnes type contour integral for I- function for one variable and then convolution of laplace transform for H-function.then solving we get required result.

Proof II First Taking the I-Function of one variable Saxena([11],1.13, 1.14) then applyLaplace transforms efine as (1.1) . Now using Gamma Function (Euler's second formula).Mellin barnes type contour integral for I- function for one variables then (using 1.11 &1.12) and then we get required result.

Proof III same as proof I & II we get required result.

3. Conclusion

Three kinds of generalized convolution operations of fractional Integral and Laplace transform are investigated, and the corresponding convolution theorems are derived, which can be seen as the generalization of the classical results. The relationships of these generalized fractional convolution operations are also discussed. Additionally, the potential applications of the derived results in solving two kinds of generalized convolution integral equations are discussed, and the explicit solutions of these convolution-type integral equations are obtained.

From this Paper we get some many solution of integral equation of convolution from having H – Function of one or more veriables

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5. Reference

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