On solution of a quadratic integral equation by Krasnoselskii theorem

Amany M. Moter

Department of Mathematic, Faculty of Basic Education, University of Kufa, Najaf, Iraq

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Abstract: in this paper, we prove the following equation

$$\alpha(\zeta) = \varphi_1(\zeta) + \Pi_1\big(\zeta, \alpha(\varphi_2(\varsigma))\big)\int_0^1 \lambda(\zeta,\varsigma)\Pi_2(\zeta,\varsigma,\alpha(\varphi_3(\varsigma)))d\varsigma, \qquad \zeta \in [0,1] \qquad (\circledast)$$

Has existence solution in the space of continuous function by Krasnoselskii theorem under of collection of assumptions.

Keyword: Krasnoselskii theorem, quadratic integral equation, Ascoli-Arzela theorem, measure of noncompactness.

i) Introduction:

The use of Integral Equations is crucial for mathematicians, engineers, and theoretical physicists in solving complex research problems. These equations are commonly utilized in dealing with initial value and boundary value problems, which may have fixed or variable boundaries. Knowledgeable individuals recognize the significant importance of Integral equations in pure and applied Mathematics. These equations play a critical role in solving physical and chemical problems. The quadratic integral equations are encountered in various fields, such as radiative transfer, neutron transport, kinetic theory of gases (see [1, 2, 5, 9, 10, 14]) and designing band-limited signals for binary communication. These equations are also used in simple memoryless correlation detection, especially when signals are disturbed by additive white Gaussian noise.

Many problems in the real world can be described using integral equations. This type of equation has been extensively studied by various authors, such as Chandrasekhar in 1947 (see [18]) and in a book published in 1960 (see [17]). These initial studies were primarily by astrophysicists, but later mathematicians also became interested in this theory. They have discovered many thought-provoking questions that still remain unanswered. Some notable contributions in this field have been made by Conti (see [6]), Banás, Argyros (see [8]), Sadarangani (see [11, 12]), Gripenberg (see [7]), Joshi [16] and numerous other.

ii) Preliminaries and Notation

We introduce notations, definitions, and theorems which are used throughout this paper.

We denoted for Banach space by $(\mathfrak{B}, \|\,.\,\|)$, $\mathfrak{B}$ is Banach space with a norm $\|\,.\,\|$, consider the set Γ is subset of $\mathfrak{B}$. as well, $\mathcal{S}_{\mathfrak{B}}$ is set of family of all bounded and empty subset of $\mathfrak{B}$**.** Thereover, we can refer to subfamily consisting of all weakly relatively compact and relatively compact sets.

Now we refer to $\mathcal{T}_{\mathfrak{B}}$ is subfamily set consisting of all weakly relatively compact and relatively compact sets.

Definition [13]: Let $\Psi: \mathcal{S}_{\mathfrak{B}} \to \mathfrak{R}^+$ is a measure of weak noncompactness in $\mathfrak{B}$ if it satisfies the following provision:

a) The family ker Ψ is nonempty and ker $\Psi \subset \mathcal{T}_{\mathfrak{B}}$ such that ker $\Psi = \{\Gamma \in \mathcal{S}_{\mathfrak{B}}: \Psi(\Gamma) = 0\}$, as well ker Ψ is called the kernel of the measure Ψ.
b) $\Gamma \subset \Upsilon \Rightarrow \Psi(\Gamma) \leq \Psi(\Upsilon)$.
c) $\Psi(conv\Gamma) = \Psi(\Gamma)$.
d) If $\Gamma_n \in \mathcal{S}_{\mathfrak{B}}, \Gamma_n = \overline{\Gamma}_n$ and the set $\Gamma_{n+1} \subset \Gamma_n$ such that $n = 1,2, \ldots$
and if $\lim_{x\to 0} \Psi(\Gamma_n) = 0$, then $\Gamma_\infty = \bigcap_{n=1}^{\infty} \Gamma_n$ is nonempty.
e) $\Psi[\rho\Gamma + (1-\rho)\Upsilon] \leq \rho\Psi(\Gamma) + (1-\rho)\Psi(\Upsilon), \rho \in [0,1]$.

The modules of continuity $\psi(\alpha, \tau)$ is defined by

$$\psi(\alpha, \tau) = \sup\{|\alpha(\lambda_1) - \alpha(\lambda_2)|: \lambda_1, \lambda_2 \in I, |\lambda_2 - \lambda_1| \leq \tau\}$$

Such that $\tau > 0$, $\alpha \in \Gamma$ and Γ is a bounded nonempty subset of continuous function on interval I.

Definition [19]: a set ς subset of $\mathfrak{B}$ is said to be convex, if all , $\rho \in [0,1]$ and all $\alpha, \beta \in \varsigma$, $\rho\,\alpha + (1-\rho)\,\beta \in \varsigma$. The following expression $\rho\,\alpha + (1-\rho)\,\beta$ is said to be a convex combination of α and β. Now we can say that if any combination of every two element in ς , it is also in ς., then the set ς is a convex set.

Theorem (Ascoli-Arzela Theorem) [3]: suppose a subset P of $C(I)$ is relatively compact if and only if the set P is bounded, and equicontinuous.

Theorem (Krasnoselskii Theorem)[15]: suppose that $\mathcal{M}$ be a nonempty, convex and closed subset of $\mathbb{E}$. Consider $\mathcal{S}$ and $\mathcal{T}$ are operator such that:

a) $\mathcal{S}(\mathcal{M}) + \mathcal{T}(\mathcal{M}) \subseteq \mathcal{M}$.

b) $\mathcal{S}$ is a contraction mapping.

c) The set $\mathcal{T}$ is continuous and the set $\mathcal{T}(\mathcal{M})$ is relatively compact.

iii) Main results

We can write the equation (⍟) as following:

$$\mathcal{S}(\alpha) = \varphi_1(\zeta) + \Pi_1\big(\zeta, \alpha(\varphi_2(\varsigma)\big) \int_0^1 \lambda(\zeta,\varsigma)\Pi_2(\zeta,\varsigma,\alpha(\varphi_3(\varsigma)))d\varsigma, \qquad \zeta \in [0,a] \qquad (⍟)$$

Such that $\mathcal{S}$ is bounded and linear operator on real number. As well as, all the hypotheses below were fulfilled:

A. The functions Π_i $\left(\Pi_i = \Pi_i\big(\zeta, \alpha(\varphi_{i+1}(\varsigma))\big)\right): [0,1] \times \mathfrak{R} \to \mathfrak{R}$ is continuous and satisfies the Lipschitz condition i.e there are $\Omega \geq 0$ such that:

$$\left|\Pi_i\left(\zeta, \alpha\big(\varphi_{i+1}(\varsigma)\big)\right) - \Pi_i\left(\zeta, \beta\big(\varphi_{i+1}(\varsigma)\big)\right)\right| \leq \Omega\left|\alpha\big(\varphi_{i+1}(\varsigma)\big) - \beta\big(\varphi_{i+1}(\varsigma)\big)\right|$$

So, we can find that:

$$\left|\Pi_i\left(\zeta, \alpha\big(\varphi_{i+1}(\varsigma)\big)\right)\right| = \left|\Pi_i\left(\zeta, \alpha\big(\varphi_{i+1}(\varsigma)\big)\right) - \Pi_i(\zeta,0) + \Pi_i(\zeta,0)\right| \leq \Omega|\alpha| + a^*$$

For each $\zeta \in [0,1]$, $\alpha, \beta \in \mathfrak{R}$ and $i = 1,2$.

B. The functions Π_i $\left(\Pi_i = \Pi_i\big(\zeta, \varsigma, \alpha(\varphi_{i+1}(\varsigma))\big)\right): [0,1] \times [0,1] \times \mathfrak{R} \to \mathfrak{R}$ is continuous and there are function define by $\mho: [0,1] \to [0,1]$ such that:

$$\left|\Pi_i\left(\zeta, \varsigma, \alpha\big(\varphi_{i+1}(\varsigma)\big)\right)\right| \leq \mho(|\alpha(\varphi_{i+1}(\varsigma))|)$$

For each $\zeta \in [0,1]$, $\alpha \in \mathfrak{R}$ and $i = 1,2$.

C. The function φ_1 is continuous and define by $\varphi_1: [0,1] \to [0,1]$.

D. The function λ $\big(\lambda = \lambda(\zeta,\varsigma)\big): [0,1] \times [0,1] \to \mathfrak{R}$ is continuous with respect to two variables ζ and ς where $\int_0^1 |\lambda(\zeta,\varsigma)|\, d\varsigma \leq a,$ such that $a > 0, \forall\, \zeta \in I$.

E. The operator $\mathcal{S}$ is bounded and linear operator on real number.

F. The following inequality is fulfilled

$$a\Omega\, \mho(\|\alpha\|) < 1$$

Theorem: If the assumption from A to F are fulfilled, then equation (⍟) has at least one solution on interval [0,1].

Proof:

- first step, let μ is positive number such that the set $\mathcal{M}_\mu$ is contain all positive function $\alpha(\zeta)$ and ζ belong to [0,1]. So $\mathcal{M}_\mu$ is nonempty and bounded set however we need prove that it is closed and convex set.

1) Suppose $\{\alpha_n\}$ is strong convergent sequence of $\mu_\circ$ and convergent to α, so there exists $\{\alpha_{k_n}\}$ subsequence of $\{\alpha_n\}$ where it is converges in interval I which means that α belong to $\mathcal{M}_\mu$. We conclude that the set $\mathcal{M}_\mu$ is closed.
2) Suppose $\alpha, \beta \in \mathcal{M}_\mu$ then $\|\alpha\| \leq \mu$ and $\|\beta\| \leq \mu$, $\rho \in [0,1]$, we get:
$\|\rho\, \alpha + (1-\rho)\, \beta\| \leq \rho\|\alpha\| + (1-\rho)\|\beta\| \leq \rho\mu + (1-\rho)\mu = \mu$, we conclude that $\mathcal{M}_\mu$ is convex.

Now, for $\alpha \in \mathcal{M}$, we have:

$$|\mathcal{S}\alpha| = \left|\varphi_1(\zeta) + \Pi_1\big(\zeta, \alpha(\varphi_2(\varsigma)\big) \int_0^1 \lambda(\zeta,\varsigma)\Pi_2(\zeta,\varsigma,\alpha(\varphi_3(\varsigma)))d\varsigma\right|$$

$$\leq |\varphi_1(\zeta)| + \left|\Pi_1\big(\zeta, \alpha(\varphi_2(\varsigma)\big)\right| \int_0^1 |\lambda(\zeta,\varsigma)|\, |\Pi_2(\zeta,\varsigma,\alpha(\varphi_3(\varsigma)))| d\varsigma$$

$$\leq \|\varphi_1\| + a\, (\Omega\|\alpha\| + a^*)\, \mho(\|\alpha\|)$$

By assumptions (C) and (E), it is imply that $\|\mathcal{S}\alpha\| \leq \mu_{\mathcal{S}}$.

Similar away, we obtain $\|\mathcal{T}\alpha\| \leq \mu_{\mathcal{T}}$. Let $\mu = \max\{\mu_{\mathcal{S}}, \mu_{\mathcal{T}}\}$ and for any $\alpha \in \mathcal{M}$

Thus,

$\|\, \mathcal{S}(\alpha) + \mathcal{T}(\alpha)\| \leq \|\mathcal{S}\alpha\| + \|\mathcal{T}\alpha\| \leq \mu$, we conclude that $\mathcal{S}(\mathcal{M}) + \mathcal{T}(\mathcal{M}) \subseteq \mathcal{M}$.

- Second step, we must prove that the operator $\mathcal{S}$ is contraction mapping.

For all $\tau > 0$ and $\alpha, \beta \in \mathfrak{B}$ where $\|\alpha - \beta\| \leq \tau$, we get:

$$|\mathcal{S}\alpha-\mathcal{S}\beta| = \Big|\varphi_1(\zeta)+\Pi_1\big(\varsigma,\alpha(\varphi_2(\varsigma)\big)\int_0^1 \lambda(\zeta,\varsigma)\Pi_2(\zeta,\varsigma,\alpha(\varphi_3(\varsigma)))d\varsigma$$

$$-\varphi_1(\zeta)-\Pi_1\big(\varsigma,\beta(\varphi_2(\varsigma)\big)\int_0^1 \lambda(\zeta,\varsigma)\Pi_2(\zeta,\varsigma,\beta(\varphi_3(\varsigma)))d\varsigma\Big|$$

$$\leq \left|\Pi_1\left(\zeta,\alpha\big(\varphi_2(\varsigma)\big)\right)\right|\int_0^1|\lambda(\zeta,\varsigma)|\left|\Pi_2\left(\zeta,\varsigma,\alpha\big(\varphi_3(\varsigma)\big)\right)-\Pi_2\left(\zeta,\varsigma,\beta\big(\varphi_3(\varsigma)\big)\right)\right|d\varsigma$$

$$+\left|\Pi_1\left(\zeta,\alpha\big(\varphi_2(\varsigma)\big)\right)-\Pi_1\left(\zeta,\beta\big(\varphi_2(\varsigma)\big)\right)\right|\int_0^1|\lambda(\zeta,\varsigma)|\left|\Pi_2\big(\zeta,\varsigma,\beta\big(\varphi_3(\varsigma)\big)\big)\right|d\varsigma$$

$$\leq a(\Omega|\alpha|+a^*)\psi(\Pi_2,\tau)+a\Omega|\alpha-\beta|\mho(|\alpha|)$$

When $\tau\to 0$ then $\psi(\Pi_2,\tau)\to 0$ and take $a\Omega\,\mho(\|\alpha\|)=\mathcal{K}$ such that $\mathcal{K}<1$, imply that

$$\|\mathcal{S}\alpha-\mathcal{S}\beta\|\leq\mathcal{K}\|\alpha-\beta\|$$

We conclude that the operator $\mathcal{S}$ is contraction mapping.

- The third step, suppose arbitrary $\zeta_1,\zeta_2\in I$ such that $\zeta_1<\zeta_2$ and $|\zeta_2-\zeta_1|\leq\tau$, we get:

$$|\mathcal{T}\alpha(\zeta_2)-\mathcal{T}\alpha(\zeta_1)| = \Big|\varphi_1(\zeta_2)+\Pi_1\big(\zeta_2,\alpha(\varphi_2(\varsigma)\big)\int_0^1 \lambda(\zeta_2,\varsigma)\Pi_2(\zeta_2,\varsigma,\alpha(\varphi_3(\varsigma)))d\varsigma$$

$$-\varphi_1(\zeta_1)+\Pi_1\big(\zeta_1,\alpha(\varphi_2(\varsigma)\big)\int_0^1 \lambda(\zeta_1,\varsigma)\Pi_2(\zeta_1,\varsigma,\alpha(\varphi_3(\varsigma)))d\varsigma\Big|$$

$$\leq|\varphi_1(\zeta_2)-\varphi_1(\zeta_1)|+\big|\Pi_1\big(\zeta_2,\alpha(\varphi_2(\varsigma)\big)\big|\int_0^1|\lambda(\zeta_2,\varsigma)|\,|\Pi_2(\zeta_2,\varsigma,\alpha(\varphi_3(\varsigma)))|d\varsigma$$

$$+\big|\Pi_1\big(\zeta_2,\alpha(\varphi_2(\varsigma)\big)-\Pi_1\big(\zeta_1,\alpha(\varphi_2(\varsigma)\big)\big|\int_0^1|\lambda(\zeta_2,\varsigma)|\,|\Pi_2(\zeta_2,\varsigma,\alpha(\varphi_3(\varsigma)))|d\varsigma$$

$$+\big|\Pi_1\big(\zeta_2,\alpha(\varphi_2(\varsigma)\big)\big|\int_0^1|\lambda(\zeta_2,\varsigma)-\lambda(\zeta_1,\varsigma)|\,|\Pi_2(\zeta_1,\varsigma,\alpha(\varphi_3(\varsigma)))|d\varsigma$$

$$+\big|\Pi_1\big(\zeta_2,\alpha(\varphi_2(\varsigma)\big)\big|\int_0^1|\lambda(\zeta_2,\varsigma)|\,|\Pi_2(\zeta_2,\varsigma,\alpha(\varphi_3(\varsigma)))-\Pi_2(\zeta_1,\varsigma,\alpha(\varphi_3(\varsigma)))|d\varsigma$$

$\leq\psi(\varphi_1,\tau)+a(\Omega\|\alpha\|+a^*)\mho(\|\alpha\|)+a\Omega\|\alpha-\beta\|\mho(\|\alpha\|)+\tau(\Omega\|\alpha\|+a^*)\mho(\|\alpha\|)+a(\Omega\|\alpha\|+a^*)\psi(\Pi_2,\tau)$

When $\tau\to 0$ then $\psi(\Pi_2,\tau),\psi(\varphi_1,\tau)\to 0$. We obtain that

$$\|\mathcal{T}\alpha(\zeta_2)-\mathcal{T}\alpha(\zeta_1)\|\leq[a\mho\mu+\tau\mho\mu](\Omega\mu+a^*)+a\Omega\mho\mu\|\alpha-\beta\|$$

So, we conclude that the operator $\mathcal{T}$ is continuous.

- ❖ The fourth step, we must prove that the operator $\mathcal{T}(\mathcal{M})$ is a relatively compact, we need show that:

a) $\mathcal{T}\alpha \in \mathcal{T}(\mathcal{M})$ is bounded, we have:

$$|\mathcal{T}\alpha| = \left|\varphi_1(\zeta) + \Pi_1\big(\zeta, \alpha(\varphi_2(\varsigma)\big) \int_0^1 \lambda(\zeta,\varsigma)\Pi_2(\zeta,\varsigma,\alpha(\varphi_3(\varsigma)))d\varsigma\right|$$

$$\leq |\varphi_1(\zeta)| + \left|\Pi_1\big(\zeta, \alpha(\varphi_2(\varsigma))\big)\right| \int_0^1 |\lambda(\zeta,\varsigma)|\,|\Pi_2(\zeta,\varsigma,\alpha(\varphi_3(\varsigma)))|d\varsigma$$

$$\leq \|\varphi_1\| + a\,(\Omega\|\alpha\| + a^*)\mho(\|\alpha\|)$$

We take $\delta = a\mho\mu(\Omega\mu + a^*)$, it is imply that $\|\mathcal{T}\alpha\| \leq \tau + \delta$. We conclude the operator $\mathcal{T}$ is bounded.

b) $\mathcal{T}\alpha \in \mathcal{T}(\mathcal{M})$ is equicontinuous, suppose arbitrary ζ_1 , $\zeta_2 \in I$ such that $\zeta_1 < \zeta_2$ and $|\zeta_2 - \zeta_1| < \tau$, by Similar way in third step we have:

$$|\mathcal{T}\alpha(\zeta_1) - \mathcal{T}\alpha(\zeta_2)| \leq \psi(\varphi_1,\tau) + [a\mho\mu + \tau\mho\mu + a\psi(\Pi_2,\tau)](\Omega\mu + a^*) + a\Omega\mho\mu\|\alpha - \beta\|$$

Since ψ is continuous function then $\psi(\varphi_1,\tau), \psi(\Pi_2,\tau) \to 0$ and $|\zeta_2 - \zeta_1| \to 0$ as $\tau \to 0$

$$|\mathcal{T}\alpha(\zeta_1) - \mathcal{T}\alpha(\zeta_2)| \leq [a\mho\mu + \tau\mho\mu](\Omega\mu + a^*) + a\Omega\mho\mu\|\alpha - \beta\|$$

We conclude the operator $\mathcal{T}\alpha$ is equicontinuous.

Now, we must demonstrate an example in which our primary finding proves to be advantageous and enables the expansion of existing theorems.

Example: let the following quadratic integral equation

$$\mathcal{S}(\alpha) = e^{\zeta} + \zeta^2 + \frac{\alpha\varsigma}{2}\int_0^1 \alpha^2\zeta^{\varsigma}\zeta cos\varsigma d\varsigma, \qquad \zeta \in [0,a] \qquad (\circledast\circledast)$$

From observing, we can conclude that above equation is a specific instance of equation ($\circledast$), so the function $\varphi_1(\zeta) = e^{\zeta}$ is continuous and $\Pi_1\big(\zeta, \alpha(\varphi_2(\varsigma)\big) = \zeta^2 + \frac{\alpha\varsigma}{2}$ where $\left|\zeta^2 + \frac{\alpha\varsigma}{2}\right| \leq \frac{1}{2}|\alpha\varsigma|$

Moreover, $\lambda(\zeta,\varsigma) = \zeta^{\varsigma}$ which implies that $|\zeta^{\varsigma}| \leq 1$ such that $\zeta,\varsigma \in I$ and the following function

$$\Pi_2(\zeta,\varsigma,\alpha(\varphi_3(\varsigma))) = \alpha^2\zeta cos\varsigma, \text{ this implies that } |\alpha^2\zeta cos\varsigma| \leq |\zeta cos\varsigma||\alpha^2|$$

Additionally, when provided with any arbitrary element $\tau > 0$ and $\alpha, \beta \in \mathfrak{B}$ where $|\alpha - \beta| \leq \tau$, we get:

$$|\alpha^2 \zeta^{\varsigma} \zeta cos\varsigma - \beta^2 \zeta^{\varsigma} \zeta cos\varsigma| \leq |\zeta^{\varsigma} \zeta cos\varsigma||\alpha^2 - \beta^2|$$

The equation (⍟⍟) is fulfilled all conditions of our theorem, so it has at least one solution on interval.

iv) Conclusion:

By collection of assumption and it is possible to locate a favorable outcome for equation(⍟), thus fulfilling all requirements for Theorem Consequently, equation (⍟) will then possess at least one feasible solution on interval [0,1].

v) Reference

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