# Dirichet And Unitary Convolutions Of Arithmetic Functions.

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**Abstract**:- In this paper we study Arithmetic functions and its distributive and quasi distributive properties hold is general set-up of s-regular and A-regular convolutions. They hold in the case of Dirichet and Unitary Convolutions. We study the theorems of Pandmavashamma [3], Vidyanatha Swamy [7], Langford [2], Lambek [1].

**(1) Introduction**:- A system study of Vasu's S-regular and A-regular i.e. strong regular convolutions has been presented along with some results by Pandmavashamma [3] Balashekhar and Subrahimanya Saitri [4].

**Theorem (1.1)** for any three arithmetic functions f, g, h, where f is A- multiplicative, and A is any s-regular convolution

**[1.1(1)]** F(g A h) = (f g) A (f h)

Holds if any only if f is completely A-multiplicative

**Theorem (1.2)** for any three A-multiplicative functions $f_1$, $f_2$, $f_3$, we have

**[1.2 (1)]** $f_1 A (f_2 B f_3) = (f_1 A f_2) B (f_1 A f_3)$, where A is any A- regular convolution and B is the unitary regular convolution associated with A.

**Theorem (1.3)** for any four completely-A-multiplicative functions $f_1$, $f_2$, $f_3$, $f_4$ we have

**[1.3 (1)]** $(f_1 f_3) A (f_1 f_4) A (f_2 f_3) A (f_3 f_4) = (f_1 A f_2) (f_3 A f_4) A \theta$ where

**[1.3 (2)]**
$$\theta(r) = \begin{cases} f_1(\sqrt{r})\, f_2(\sqrt{r})\, f_3(\sqrt{r})\, f_4(\sqrt{r}), & \text{if r is a square w. r. to product representation of A} \\ 0, & \text{otherwise.} \end{cases}$$

We also indicate the corresponding distributive properties for six completely-A-multiplicative functions, an analogue of Theorem of Subbarao [5].

**Theorem (1.4)** For completely-A-multiplicative function f, g, h, k, u, v, we have

**[1.4 (1)]** (f A g) (h A k) (u A v)=(f hu A f hu A f ku A f kv A ghu A ghu A gku A gkv A θ).

Where θ (n) is an A-multiplicative function defined for an arbitrary A-primitive $p^t$ as follows:

**[1.4 (2)]**

$$\theta(p^{t_i}) = \begin{cases} 1, & \text{if } i = 0 \\ 0, & \text{if } i = 1 \text{ or } i = 5, \rho(p^t) > 4, \text{ or } i > 6, \rho(p^t) > 6 \\ -w, & \text{if } i = 2, \rho(p^t) \geq 2 \\ 2\,abcdrs\,(a+b)(c+d)(r+s), & \text{if } i = 3, \rho(p^t) \geq 3 \\ -2\,abcdrsw, & \text{if } i = 4, \rho(p^t) > 4 \\ (abcdrs)^3, & \text{if } i = 6, \rho(p^t) \geq 6 \end{cases}$$

Where $f(p^t) = a$ ; $g(p^t) = b$ ; $h(p^t) = c$ ; $k(p^t) = d$ ; $u(p^t) = r$ ; $b(p^t) = s$

**2.** We make use of the generating power series and Bell series defined below:

$$f_{p^t}(x) = \sum_{i=0}^{\rho} f\left(P^{t_i}\right) x^i$$

is called the generating power series or Bell series to the base $P^t$.

If f is A-multiplicative, f is uniquely determined when $f_{p^t}(x)$ are known for all A-primitives $p^t$.

**Example (2.1):** Mobius function $\mu_A = u_A^{-1}$ (associated with an A-divisor system), where u (n) = 1 for all n is determined by the generating power series.

**[2.1 (1)]** $\mu_{p^t}(x) = 1-x.$

**Result (2.2):** If f is completed-A-multiplicative, then $f(p^{t_i}) = f(p^t)^i$, for $0 \leq i \leq \rho(p^t)$

Where $p^t$ is an A-primitive and its generating power series to the base $p^t$ is

**[2.2 (2)]** $f_{p^t}(x) = 1\text{-}f(p^t)\,x)^{-1} \pmod{x^{\rho+1}} = 1\text{-}f(p^t)\,x)^{-1}|_{\rho}$

the binomial series $(1\text{-}f(p^t)\,x)^{-1}$ truncated to degree ρ. Likewise we have

**Result (2.3):** If f and g are A-multiplicative then

**[2.3 (1)]** $(fAg)_{p^t}(x) = f_{p^t}(x) \overset{A}{x} g_{p^t}(x)$

Where $\overset{A}{x}$ indicates that the product is the formal product of the concerned power series modulo

$x^{\rho+1}$. We also have

**Result (2.4):** If $f_1$ and $f_2$ are completely -A-multiplicative and $f_1(p^t) = a$, $f_2(p^t) = b$ then

**[2.4 (1)]** $(f_1 f_2)_{p^t}(x) = [1 - abx]^{-1}|_\rho$

We know

**Result (2.5):** $$\sum_{n=0}^{\infty}\left[\frac{a^{n+1} - b^{n+1}}{a-b}\right]\left[\frac{c^{n+1} - d^{n+1}}{c-d}\right]x^n$$

$$= \frac{(1-a\,b\,c\,d\,x^2)}{(1-a\,c\,x)\,(1-bcx)(1-\,a\,d\,x)(1-b\,d\,x)}$$ Sivaramakrishnan [6].

Hence on truncating to the degree $x^\rho$, we have

**Result (2.6):** $$\sum_{n=0}^{\infty}\left\{\left[\frac{a^{n+1} - b^{n+1}}{a-b}\right]\left[\frac{c^{n+1} - d^{n+1}}{c-d}\right]x^n\right\}|_\rho$$

$$= \frac{(1-a\,b\,c\,d\,x^2)}{(1-a\,c\,x)(1-\,b\,c\,x)(1-a\,d\,x)(1-b\,d\,x)}\ |_\rho$$

**3.** We all prove theorem (1.1) in a more general form.

**Definition (3.1):** A product k= g A h for any two arithmetic functions g, h is called a discriminative A-product if k (n) = g (1) h (n) + g (n) h (1) holds only when n is A-primitive.

In otherwords, k = g A h is a discriminative –A-product only if

**[3.1 (1)** $$\sum_{\substack{d\,\varepsilon\,A(n)\\ d\neq 1, d\neq n}} g\,(d)h\left(\frac{n}{d}\right) \neq 0 \text{ when n is not A} - \text{primitive}$$

**Theorem (3.2):** Let f, g, h be any three arithmetic functions, then f is completely-A- multiplicative if and only if [1.1 (1)] holds for a discriminative -A-product k = gAh.

**Proof:** It is easy to verify that [1.1 (1)] holds for all arithmetic functions g, h whenever f is completely -A-multiplicative. In particular, it holds when k = gAh, a discriminative A-product.

For the converse, we assume $r = p_1^{t_1}\ p_2^{t_2}\ .\ .\ .\ p_m^{t_m}$, where the A-primitives $p_i^{t_i}$ are distinct or not.

We prove by induction on m, the necessary result for completely-A-multiplicatively, viz.,

**[3.2 (1)]** $$f\left(\prod_{i=1}^{m} p_i^{t_i}\right) = \prod_{i=1}^{m} f\left(p_i^{t_i}\right)$$

If m = 1, (3.2) is trivial.

Assume (3.2) is true for all m < n. Let gAh be a discriminative

A-product and m = n. Then, since f satisfies [1.1(2)],

$$f(p_1^{t_1}\ p_2^{t_2}\ldots p_n^{t_n})\sum_{d\,\varepsilon\,A(r)} g(d)h\left(\frac{r}{d}\right) = \sum_{d\,\varepsilon\,A(r)} f(d)g(d)f\left(\frac{r}{d}\right) g\left(\frac{r}{d}\right)$$

$$= f(p_1^{t_1})\, f(p_m^{t_m}) \sum_{\substack{d\,\varepsilon\,A(n)\\ d\neq 1, d\neq n}} g(d) + f(p_1^{t_1}\ldots p_n^{t_n},)\, h(p_1^{t_1}\ldots p_n^{t_n})\, g(1)$$

$$+ f(p_1^{t_1}\ldots p_n^{t_n})\, h(1)\, g(p_1^{t_1}\ldots p_n^{t_n})$$

Therefore

$$\left[f(p_1^{t_1}\ldots p_n^{t_n}) - f(p_1^{t_1})\ldots\ldots f(p_n^{t_n})\right] \sum_{\substack{d\,\varepsilon\,A(n)\\ d\neq 1, d\neq n}} g(d)\, h\left(\frac{n}{d}\right) = 0,$$

So, That (3.2) holds for m = n, as the product g A h is discriminative.

In particular, the converse is true, when [1.1 (1)] holds for all arithmetic functions f, g, h.

**Proof of the Theorem (1.2):** It is enough if we that the result is true for $p^{i\,t}$, $1\leq i \leq \rho$ for any A-primitive $p^t$. Note that for any A-regular convolution there exist a corresponding unique "unitary" s-regular convolution B. In fact

$$(f_2\, B\, f_3)\, p^{it}) = \begin{cases} 1 & \text{if } i = 0 \\ f_2(p^{it}) + f_3(p^{it}), & \text{if } i \geq 1 \end{cases}$$

$$\therefore \quad (f_1\, A\, (f_2\, B\, f_3)\, (p^{it}) = \sum_{1\leq j+\ell\leq i} f\, (p^{jt})\, f_2(p^{\ell t}) + f_3(p^{\ell t}) + f_1(p^{it})$$

$$(f_1\, A f_k)\, (p^{it}) = \sum_{0\leq j+\ell\leq i} f_1(p^{jt})\, f_k(p^{\ell t}),\ k = 2, 3$$

Therefore

$$(f_1\, A\, (f_2\, B\, f_3)\, B\, f_1)\ (p^{it}) = (f_1\, A\, f_3)\, (p^{it}).$$

Hence

$$f_1\, A\, (f_2\, B\, f_3)\, B\, f_1 = (f_1\, A\, f_2)\, B\, (f_1\, A\, f_3)$$

**Proof of the theorem (1.3):** Consider any primitive base $p^t$.

Let $a_i = f_1(p^t)$, i = 1, 2, 3, 4,

$$a_1 \neq a_2\,;\ a_3 \neq a_4$$

Then for any i , j = 1, 2, 3, 4 $i \neq j$ we have,

$$\left(f_1\, A\, f_j\right)_{p^t}(x) = \sum_{\ell=0}^{\rho}\left[\frac{a_i^{\ell+1} - a_j^{\ell+1}}{a_i - a_j}\right] x^{\ell} \; \ldots \ldots .$$

and

$$(f_i\;\; f_i)_{p^t}(x) = 1 + \sum_{\ell=1}^{\rho} a_i^{\ell}\;\; a_i^{\ell}\;\; x^{\ell}$$

Then if f = $(f_1 \, A \, f_2)\, (f_3 \, A \, f_4)$ in view of, [2.4 (1)] we have

$$F_{p^t}(x) = (1\text{-}a_1\; a_2\, a_3\, a_4\;\; x^2)\, (1\text{-}\, a_1\;\; a_3\, x)^{-1}\, (1\text{-}a_2\;\; a_3)^{-1}$$

$$(1\text{-}\, a_1\;\; a_4\, x)^{-1}\;\; (1\text{-}a_2\;\; a_4\, x)^{-1}$$

Further, we will have

$$\theta_{p^t}(x) = 1\text{-}a_1\;\; a_2\;\; a_3\;\; a_4\, x^2 + \ldots\ldots + a_1^{\ell}\;\; a_2^{\ell}\, a_3^{\ell}\;\; a_4^{\ell}\, x^{2\ell}$$

$$= (1\text{-}\, a_1\;\; a_2\;\; a_3\;\; a_4\, x^2)^{-1}|_{\rho}$$

Where $\ell = [\rho\, (p^t)\, /2]$.

Hence Theorem (1.3) follows:

**Proof of Theorem (1.4):** It follows on the same lines, using generating power series to bases $p^t$ modulo $x^{\rho+1}$ and on verifying that left side of (1.9) has the generating series to base $p^t$ given by

$$[\, (1 - acrx\, )^{-1}\, (1 - acsx\, )^{-1}\, (\, 1\text{-}\, adrx\, )^{-1}(\, 1\text{-}\, adsx\, )^{-1}\, (\, 1\text{-}\, bcrx\, )^{-1}(1 - bcsx\, )^{-1}\, (1 - bdrx\, )^{-1}$$

$$(1 - bdsx\, )^{-1}]\, |_{\rho}$$

$$= [1 - wx^2 + 2\, abcdrs\, (a+b)\, (c+d)\, (r+s)\, x^3 - abcdrswx^4 - (abcdrs)^3\, x^6]\, |_{\rho}$$

$$= \theta_{p^t}(x).$$

<u>**Conclusion**</u>: The proof Theorem (1.1), (1.2), (1.3) and (1.4) follows with the use of generating power series and Bell series so that (3.2) holds for m=n as the product of g A h is discriminative. In particular the converse is true when [1.1 (1)] holds for all arithmetic function f, g, h.

<u>**Reference**</u>

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