GENERALIZED FRACTIONAL DERIVATIVE OPERATORS OF THE PRODUCT OF MULTI-INDEX BESSEL FUNCTION AND MULTI-INDEX MITTAG LEFFLER FUNCTION WITH APPLICATIONS

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Abstract

In the present paper we have introduced multi-index Bessel function and multi-index Mittag Leffler function and established the generalized fractional derivative operators of the product of generalized multi-index Bessel function and multi-index Mittag Leffler function. Further, the Riemann-Liouville, fractional derivative operators of given functions are obtained.
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Definitions

1 Generalized Multi-Index Bessel Function

For $A_j, B_j, \lambda, \in \mathbb{C}, \mu > 0, \mathcal{R}e(\lambda) > 0$ the generalized multi-index Bessel function is defined by Choi and Agarwal [1] in the following summation form:

$$\mathrm{J}_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x) = \sum_{l=0}^{\infty} \frac{(\lambda)_{\mu l}}{\prod_{j=1}^{m} \Gamma(A_j\, l + B_j + 1)} \frac{(-x)^l}{l!}; \quad (m \in \mathbb{N}) \tag{1.1}$$

where $\mathcal{R}e(B_j) > -1$ and $\sum_{j=1}^{m} \mathcal{R}e(A_j) > max\,\{0;\ \mathcal{R}e(\mu) - 1\}$

2 Generalized Multi-Index Mittag Leffler Function

For $A_j, B_j, \lambda, \rho \in \mathbb{C}$, the generalized multi-index Mittag Leffler function is defined by Saxena and Nishimoto [13] in the following summation form

$$E_{(A_j,B_j)_m}^{\lambda,\rho}(x) = \sum_{k=0}^{\infty} \frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m} \Gamma(A_j\, k + B_j)} \frac{x^k}{k!}; \quad (m \in \mathbb{N}) \tag{2.1}$$

where $\mathcal{R}e(B_j) > 0$ and $\sum_{j=1}^{m} \mathcal{R}e(A_j) > max\,\{0;\ \mathcal{R}e(\rho) - 1; 0\}$.

For $m = 1$ the generalized multi-index Mittag Leffler function (2.1) reduce into the generalized Mittag-Leffler function given by Shukla and Prajapati [16] and defined as

$$E_{A,B}^{\lambda,\rho}(x) = \sum_{k=0}^{\infty} \frac{(\lambda)_{\rho k}}{\Gamma(Ak + B)} \frac{x^k}{k!}, \tag{2.2}$$

where $A, B, \lambda \in \mathbb{C};\ \mathcal{R}e(A) > 0, \mathcal{R}e(B) > 0, \mathcal{R}e(\lambda) > 0$ and $\rho \in (0,1) \cup \mathbb{N}$

For $m = 1$ and $\rho = 1$, the generalized multi-index Mittag Leffler function (2.1) reduce into the_generalized Mittag-Leffler function given by Prabhakar [10] defined as

$$E_{A,B}^{\lambda}(x) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{\Gamma(Ak+B)} \frac{x^k}{k!}, \tag{2.3}$$

where $A, B, \lambda \in \mathbb{C}$; $\mathcal{Re}(A) > 0$, $\mathcal{Re}(B) > 0$, $\mathcal{Re}(\lambda) > 0, x \in \mathbb{C}$ and $(\lambda)_k$ is the well known Pochhammer symbol.

3 Generalized Fractional Derivative Operators

For $\mu = [\mathrm{Re}(\gamma)+1]$ and $x > 0$, where $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \in \mathbb{C}$ with $\mathrm{Re}(\gamma) > 0$, the generalized fractional derivative operators are defined [12, 15] as follows:

$$D_{0+}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma} f(x) = (I_{0+}^{-\alpha_2,-\alpha_1,-\beta_2,-\beta_1,-\gamma} f)(x); \quad \mathrm{Re}(\gamma) > 0$$

$$= \left(\frac{d}{dx}\right)^{\mu} (I_{0+}^{-\alpha_2,-\alpha_1,-\beta_2+\mu,-\beta_1,-\gamma+\mu} f)(x). \tag{3.1}$$

$$D_{0-}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma} f(x) = (I_{0-}^{-\alpha_2,-\alpha_1,-\beta_2,-\beta_1,-\gamma} f)(x); \quad \mathrm{Re}(\gamma) > 0$$

$$= \left(-\frac{d}{dx}\right)^{\mu} (I_{0-}^{-\alpha_2,-\alpha_1,-\beta_2,-\beta_1+\mu,-\gamma+\mu} f)(x). \tag{3.2}$$

Where generalized fractional integral operators are defined as

$$(I_{0+}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma} f)(x)$$

$$= \frac{x^{-\alpha_1}}{\Gamma(\gamma)} \int_0^x \frac{(x-t)^{\gamma-1}}{t^{\alpha_2}} F_3\left(\alpha_1, \alpha_2, \beta_1, \beta_2; \gamma; 1-\frac{t}{x}, 1-\frac{x}{t}\right) f(t)dt$$

and

$$(I_{0-}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma} f)(x)$$

$$= \frac{x^{-\alpha_2}}{\Gamma(\gamma)} \int_x^{\infty} \frac{(t-x)^{\gamma-1}}{t^{\alpha_1}} F_3\left(\alpha_1, \alpha_2, \beta_1, \beta_2; \gamma; 1-\frac{x}{t}, 1-\frac{t}{x},\right) f(t)dt.$$

The following image formula for a power function under the generalized fractional integral operators is given [15] as follows:

$$(I_{0+}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma} t^{\varepsilon-1})(x)$$

$$= \Gamma\begin{bmatrix} \varepsilon, & \varepsilon+\gamma-\alpha_1-\alpha_2-\beta_1, & \varepsilon-\alpha_2+\beta_2 \\ \varepsilon+\beta_2, & \varepsilon+\gamma-\alpha_1-\alpha_2, & \varepsilon+\gamma-\alpha_2-\beta_1, \end{bmatrix} x^{\varepsilon-\alpha_1-\alpha_2+\gamma-1}, \tag{3.3}$$

where $\mathrm{Re}(\varepsilon) > max\{\mathrm{Re}(\alpha_1+\alpha_2+\beta_1-\delta), \mathrm{Re}(\alpha_2-\beta_2), 0\}$ and

$$(I_{0-}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma} t^{-\varepsilon})(x)$$

$$= \Gamma\begin{bmatrix} \alpha_1+\alpha_2-\gamma+\varepsilon, & \alpha_1+\beta_2-\gamma+\varepsilon, & -\beta_1+\varepsilon \\ \varepsilon, & \alpha_1+\alpha_2+\beta_2-\gamma+\varepsilon, & \alpha_1-\beta_1+\varepsilon \end{bmatrix} x^{-\varepsilon-\alpha_1-\alpha_2+\gamma}, \tag{3.4}$$

where $\mathrm{Re}(\varepsilon) < 1 + min\{\mathrm{Re}(-\beta_1), \mathrm{Re}(\alpha_1+\beta_2-\gamma), \mathrm{Re}(\alpha_1+\alpha_2-\gamma)\}$ and

$$\Gamma\begin{bmatrix}\alpha_1, & \alpha_2, & \alpha_3\\ \beta_1, & \beta_2, & \beta_3\end{bmatrix} = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)\Gamma(\alpha_3)}{\Gamma(\beta_1)\Gamma(\beta_2)\Gamma(\beta_3)}$$

The generalized fractional derivative operators reduce into the saigo fractional derivative operators due to the following relations:

$$(D_{0+}^{0,\alpha_2,\beta_1,\beta_2,\gamma} f)(x) = (D_{0+}^{\gamma,\alpha_2-\gamma,\beta_2-\gamma} f)(x);\ \mathrm{Re}(\gamma) > 0 \quad (3.5)$$

$$(D_{0-}^{0,\alpha_2,\beta_1,\beta_2,\gamma} f)(x) = (D_{0,\infty}^{\gamma,\alpha_2-\gamma,\beta_2-\gamma} f)(x);\ \mathrm{Re}(\gamma) > 0 \quad (3.6)$$

$$D_{0+}^{\alpha_1,\beta_1,\gamma} f(x) = (I_{0+}^{-\alpha_1,-\beta_1,\alpha_1+\gamma} f)(x);\ \ \mathrm{Re}(\alpha_1) > 0$$

$$= \left(\frac{d}{dx}\right)^{\mu} (I_{0+}^{-\alpha_1+\mu,-\beta_1-\mu,\alpha_1+\gamma-\mu} f)(x);\ \mu = [\mathrm{Re}(\alpha_1) + 1]. \quad (3.7)$$

$$D_{0-}^{\alpha_1,\beta_1,\gamma} f(x) = (I_{0-}^{-\alpha_1,-\beta_1,\alpha_1+\gamma} f)(x);\ \ \mathrm{Re}(\alpha_1) > 0$$

$$= \left(-\frac{d}{dx}\right)^{\mu} (I_{0-}^{-\alpha_1+\mu,-\beta_1-\mu,\alpha_1+\gamma} f)(x); \mu = [\mathrm{Re}(\alpha_1) + 1], \quad (3.8)$$

Where saigo fractional integral operators introduced by saigo′s [11] and defined as follows:

$$(I_{0+}^{\alpha_1,\beta_1,\gamma} f)(x)$$

$$= \frac{x^{-\alpha_1-\beta_1}}{\Gamma(\alpha_1)} \int_0^x (x-t)^{\alpha_1-1}\, {}_2F_1\left(\alpha_1+\beta_1, -\gamma;\ \alpha_1; 1-\frac{t}{x}\right) f(t)dt,$$

$$(I_{0-}^{\alpha_1,\beta_1,\gamma} f)(x)$$

$$= \frac{1}{\Gamma(\alpha_1)} \int_x^\infty (t-x)^{\alpha_1-1}\, t^{-\alpha_1-\beta_1}\, {}_2F_1\left(\alpha_1+\beta_1, -\gamma;\ \alpha_1; 1-\frac{x}{t}\right) f(t)dt.$$

The following image formula for a power function under the saigo′s fractional integral operators is given [12] as follows:

$$(I_{0+}^{\alpha_1,\beta_1,\gamma}\, t^{\varepsilon-1})(x) = \frac{\Gamma(\varepsilon)\Gamma(\varepsilon+\gamma-\beta_1)}{\Gamma(\varepsilon+\alpha_1+\gamma)\Gamma(\varepsilon-\beta_1)} x^{\varepsilon-\beta_1-1} \quad (3.9)$$

where $\mathrm{Re}(\varepsilon) > max\{0, \mathrm{Re}(\beta_1-\gamma)\}$ and

$$(I_{0-}^{\alpha_1,\beta_1,\gamma}\, t^{\varepsilon-1})(x) = \frac{\Gamma(\beta_1-\varepsilon+1)\Gamma(\gamma-\varepsilon+1)}{\Gamma(1-\varepsilon)\Gamma(\alpha_1+\beta_1+\gamma-\varepsilon+1)} x^{\varepsilon-\beta_1-1} \quad (3.10)$$

where $\mathrm{Re}(\varepsilon) < 1 + min\{\mathrm{Re}(\beta_1),\ \mathrm{Re}(\gamma)\}$.

Let $f_1(x) = \sum_{k=0}^{\infty} C_k x^k$ and $f_2(x) = \sum_{k=0}^{\infty} D_k x^k$ be two analytic fuctions with their radii of convergence R_{f_1} and R_{f_2}, respectively. Then their Hadamard product [9] is given by the following power series:

$$f_1 * f_2(x) = f_2 * f_1(x) = \sum_{k=0}^{\infty} C_k D_k x^k;\ \ (|x| < R), \quad (3.11)$$

Where $R_c \geq R_{f_1}.R_{f_2}$ is the radius of convergence of the composite series.

4. Main Results

Theorem 1. Let $\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \gamma \in \mathbb{C}$ be such that $x > 0, \mathrm{Re}(\gamma) > 0$ and the conditions given in (1 . 1) , (2 . 1) and (3 . 1) be satisfied. Then the left sided generalized fractional derivative of the product of generalized multi-index Bessel function $J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x)$ and multi-index Mittag Leffler function $E_{(A_j,B_j)_m}^{\lambda,\rho}(x)$ is given by

$$[\,D_{0+}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma}\{t^{\delta-1}J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1 t)\ \times\ E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2 t)\}](x)$$

$$= x^{\delta+\alpha_1+\alpha_2-\gamma-1}\left\{J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1)\ \times\ E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2)\right\}$$

$$\otimes\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(l+k+\delta)\Gamma(l+k+\delta-\gamma+\alpha_1+\alpha_2+\beta_2)\Gamma(l+k+\delta+\alpha_1-\beta_1)}{\Gamma(l+k+\delta-\beta_1)\Gamma(l+k+\delta-\gamma+\alpha_1+\alpha_2)\Gamma(l+k+\delta-\gamma+\alpha_1+\beta_2)}x^{l+k}$$

(4.1)

Where $\otimes$ stands for convolution product of two functions

Proof. We refer to the left hand side of equation (4.1) by the symbol D_1.

Then making the use of equation (1.1),(2.1)and (3.1) in (4.1), we have

$$D_1 \equiv$$

$$\left[D_{0+}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma}\ \left\{t^{\delta-1}\sum_{l=0}^{\infty}\frac{(\lambda)_{\mu l}}{\prod_{j=1}^{m}\Gamma(A_j\,l+B_j+1)}\frac{(-x_1 t)^l}{l!}\ \times\sum_{k=0}^{\infty}\frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\,k+B_j)}\frac{(x_2 t)^k}{k!}\right\}\right](x)$$

After changing the order of summations and derivative operator under the conditions of theorem, we obtain the above as

$$=\sum_{l=0}^{\infty}\frac{(\lambda)_{\mu l}}{\prod_{j=1}^{m}\Gamma(A_j\,l+B_j+1)}\frac{(-x_1)^l}{l!}\times\sum_{k=0}^{\infty}\frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\,k+B_j)}\frac{(x_2)^k}{k!}$$

$$\times\,[D_{0+}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma}(t^{\delta+k+l-1})](x)$$

$$=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\lambda)_{\mu l}(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\,l+B_j+1)\times\prod_{j=1}^{m}\Gamma(A_j\,k+B_j)}\frac{(-x_1)^l}{l!}\frac{(x_2)^k}{k!}$$

$$\times [I_{0+}^{-\alpha_2,-\alpha_1,-\beta_2,-\beta_1,-\gamma}(t^{\delta+k+l-1})](x).$$

Using the image formula for power function under generalized operator (3.3),we get

$$= \sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\lambda)_{\mu l}(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\,l+B_j+1)\times\prod_{j=1}^{m}\Gamma(A_j\,k+B_j)}$$

$$\times\frac{(-x_1)^l}{l!}\frac{(x_2)^k}{k!}x^{\delta+k+l+\alpha_1+\alpha_2-\gamma-1}$$

$$\frac{\Gamma(k+l+\delta)\Gamma(\alpha_1+\alpha_2+\beta_2-\gamma+\delta+k+l)\Gamma(\alpha_1-\beta_1+\delta+k+l)}{\Gamma(-\beta_1+\delta+k+l)\Gamma(\alpha_1+\alpha_2-\gamma+\delta+k+l)\Gamma(\alpha_1+\beta_2-\gamma+\delta+k+l)}$$

Further, applying the definition (1.1) and (2.1) and convolution product on two series, we obtain

$$D_1 \equiv x^{\delta+\alpha_1+\alpha_2-\gamma-1}\left\{J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1)\times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2)\right\}$$

$$\otimes\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(l+k+\delta)\Gamma(l+k+\delta-\gamma+\alpha_1+\alpha_2+\beta_2)\Gamma(l+k+\delta+\alpha_1-\beta_1)}{\Gamma(l+k+\delta-\beta_1)\Gamma(l+k+\delta-\gamma+\alpha_1+\alpha_2)\Gamma(l+k+\delta-\gamma+\alpha_1+\beta_2)}x^{l+k}$$

Where $\otimes$ stands for convolution product of two functions

Theorem 2 Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma\in\mathbb{C}$ be such that $x>0, \mathrm{Re}(\gamma)>0$ and the Conditions given in (1.1), (2.1) and (3.2) be satisfied. Then the right sided generalized fractional derivative of the product of generalized multi-index Bessel function $J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x)$ and multi-index Mittag Leffler function $E_{(A_j,B_j)_m}^{\lambda,\rho}(x)$ is given by

$$[D_{0-}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma}\{t^{-\delta}J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1t)\times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2t)\}](x)$$

$$= x^{-\delta+\alpha_1+\alpha_2-\gamma}\left\{J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1)\times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2)\right\}$$

$$\otimes\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\alpha_1-\alpha_2+\gamma+\delta-l-k)\Gamma(-\alpha_2-\beta_1+\gamma+\delta-l-k)\Gamma(\beta_2+\delta-l-k)}{(\delta-l-k)\Gamma(-\alpha_1-\alpha_2-\beta_1+\gamma+\delta-l-k)\Gamma(-\alpha_2+\beta_2+\delta-l-k}x^{l+k}. \tag{4.2}$$

Where $\otimes$ stands for convolution product of two functions

Proof. We refer to the left hand side of equation (4.2) by the symbol D_2.

Then making the use of equation (1.1),(2.1)and (3.2) in (4.2), we have

$$D_2 \equiv \left[D_{0-}^{\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma}\left\{t^{-\delta}\sum_{l=0}^{\infty}\frac{(\lambda)_{\mu l}}{\prod_{j=1}^{m}\Gamma(A_j\, l+B_j+1)}\frac{(-x_1 t)^l}{l!}\times\sum_{k=0}^{\infty}\frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\, k+B_j)}\frac{(x_2 t)^k}{k!}\right\}\right](x)$$

After changing the order of summations and derivative operator under the conditions of

theorem, we obtain the above as

$$=\sum_{l=0}^{\infty}\frac{(\lambda)_{\mu l}}{\prod_{j=1}^{m}\Gamma(A_j\, l+B_j+1)}\frac{(-x_1)^l}{l!}\times\sum_{k=0}^{\infty}\frac{(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\, k+B_j)}\frac{(x_2)^k}{k!}$$

$$\times\,[D_{0-}^{\alpha_1,\,\alpha_2,\,\beta_1,\beta_2,\gamma}(t^{-\delta+l+k})](x).$$

$$=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\lambda)_{\mu l}(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\, l+B_j+1)\times\prod_{j=1}^{m}\Gamma(A_j\, k+B_j)}\frac{(-x_1)^l}{l!}\frac{(x_2)^k}{k!}$$

$$\times\,[I_{0-}^{-\alpha_2,-\alpha_1,-\beta_2,-\beta_1,-\gamma}(t^{-\delta+l+k})](x)$$

Using the image formula for power function under generalized operator (3.4),we get

$$=\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\lambda)_{\mu l}(\lambda)_{\rho k}}{\prod_{j=1}^{m}\Gamma(A_j\, l+B_j+1)\times\prod_{j=1}^{m}\Gamma(A_j\, k+B_j)}$$

$$\times\frac{(-x_1)^l}{l!}\frac{(x_2)^k}{k!}x^{-\delta+l+k+\alpha_1+\alpha_2-\gamma}$$

$$\frac{\Gamma(-\alpha_1-\alpha_2+\gamma+\delta-l-k)\Gamma(-\alpha_2-\beta_1+\gamma+\delta-l-k)\Gamma(\beta_2+\delta-l-k)}{\Gamma(\delta-l-k)\Gamma(-\alpha_1-\alpha_2-\beta_1+\gamma+\delta-l-k)\Gamma(-\alpha_2+\beta_2+\delta-l-k)}$$

Further, applying the definition (1.1) and (2.1) and convolution product on two series, we obtain

$$D_2 \equiv x^{-\delta+\alpha_1+\alpha_2-\gamma}\left\{J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1)\times E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2)\right\}$$

$$\otimes\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(-\alpha_1-\alpha_2+\gamma+\delta-l-k)\Gamma(-\alpha_2-\beta_1+\gamma+\delta-l-k)\Gamma(\beta_2+\delta-l-k)}{\Gamma(\delta-l-k)\Gamma(-\alpha_1-\alpha_2-\beta_1+\gamma+\delta-l-k)\Gamma(-\alpha_2+\beta_2+\delta-l-k)}x^{l+k}$$

Where ⊗ stands for convolution product of two functions

5 . Special cases of the main results.

Corollary 1. Let the conditions of Theorem 1 be satisfied and $\alpha_1=0$, $\beta_1=1$ then the Theorem 1 reduced in the following form:

$$[\,D_{0+}^{0,\alpha_2,1,\beta_2,\gamma}\{t^{\delta-1}J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1t)\ \times\ E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2t)\}](x)$$

$$=x^{\delta+\alpha_2-\gamma-1}\left\{J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1)\ \times\ E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2)\right\}$$

$$\otimes\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(l+k+\delta)\Gamma(l+k+\delta-\gamma+\alpha_2+\beta_2)}{\Gamma(l+k+\delta-\gamma+\alpha_2)\Gamma(l+k+\delta-\gamma+\beta_2)}x^{l+k}$$

(5.1)

Where ⊗ stands for convolution product of two functions

Corollary 2. Let the conditions of Theorem 2 be satisfied and $\alpha_2=0$, $\beta_2=1$ then the Theorem 2 reduced in the following form:

$$[\,D_{0-}^{\alpha_1,0,\beta_1,1,\gamma}\{t^{-\delta}J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1t)\ \times\ E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2t)\}](x)$$

$$=x^{-\delta+\alpha_1-\gamma}\left\{J_{(B_j)_{m,\mu}}^{(A_j)_{m,\lambda}}(x_1)\ \times\ E_{(A_j,B_j)_m}^{\lambda,\rho}(x_2)\right\}$$

$$\otimes\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\frac{\Gamma(-\alpha_1+\gamma+\delta-l-k)\Gamma(-\beta_1+\gamma+\delta-l-k)}{\Gamma(\delta-l-k)\Gamma(-\alpha_1-\beta_1+\gamma+\delta-l-k)}x^{l+k}.$$

(5.2)

Where ⊗ stands for convolution product of two functions

REFERENCES

[1] **Choi, J. and Agarwal, P. :** A note on fractional integral operator associated with multi-index Mittag-leffler functions, Filomat, Vol. 30(7) (2016):1931-1939.

[2] **Choi, J. Kumar, D. and Purohit, S. D.:** Integral formulas involving a product of generalized Bessel functions of the first kind, Kyungpook Math. J. Vol.56(1) (2016):131-136.

[3] **Erdelye, A. et al.:** Higher Transcendental Fucntions,Vol. I, McGraw-Hill, New York London (1953).

[4] **Kilbas, A. A. and Sebastian, N.:** Generalized fractional integration of Bessel Function of first kind, Integral Transform Special Function, Vol. 19(2008):869-883.

[5] **Kataria, K. K. and Vellaisamy, P.:** The k-Wright function and Marichev-saigo-Maeda fractional operators, J. Anal. Vol. 23(2015):75-87.

[6] **Lavoie, J. L. and Trottier, G.:** On the sum of certain Appell's series, Ganita (1969):2043-46

[7] **Mishra, V. N. Suthar, D. L. and Purohit, S.D.:** Marichev-Saigo-Maeda fractional calculus operators, Srivastava polynomials and generalized mittag-Leffler function, Cogent mathematics 4(2017)

[8] **Oberhettinger, F**.: Tables of Mellin Transforms, Springer, New York(1947).

[9] **Pohlen, T.:** The Hadamard product and universal power series, Ph.D. Thesis, Universitat Trier, Trier, Germany(2009).

[10] **Prabhakar, T. R.:** A singular integral equation with a generalized Mittag Leffler function in the kernel, Yokohama Math. J.Vol.19(1971):7-15.

[11] **Saigo, M.:** A remark on integral operators involving the Gauss hypergeometric functions. Math. Rep. College General Ed Kyushu univ., Vol.11(1978):135-143.

[12] **Saigo, M. and Maeda, N.:** More generalization of fractional calculus, transform Method and Special Functions., Bulgarian Acadmey of Sciences, Sofia Vol. 96(1998):386-400.

[13] **Saxena, R.K. and Nishimoto, K.:** N-Fractional calculus of generalized Mittag-Leffler functions, J. Frac. Calc., Vol. 37(2010):43-52.

[14] **Saxena, R.K. and Saigo, M.:** Generalized fractional calculus of the H-function associated with the Appell function F_3, Journal of Fractional calculus, Vol. 19(2001): 89-104.

[15] **Saxena, R.K. and Parmar, R.K.:** Fractional intergration and differention of the generalized Mathieu series, axioms 6(2017):18.

[16] **Shukla, A. K. and Prajapati, J.C.:** On a generalization of Mittag Leffler function and itsproperties, J. Math. Ann. Appl.,Vol. 336(2)(2007): 797-811.

[17] **Srivastava, H.M.:** A contour integral involving Fox's H-function, India J. Math. Vol.14(1972):1-6

[18] **Wright, E. M.:** The asymptotic expansion of the generalized hypergeometric function ll, Proc. London Math. Soc. Vol. 46(2) (1940):389-408.