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Mittag Leffler Function React as Conditional Probability Function

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Abstract - *The Mittag-Leffler function is a mathematical tool used in many different domains to describe complex dynamics. It emphasizes how this function can be applied to probability studies as a conditional probability distribution function (CPDF). Researchers can model conditional probabilities and examine stochastic systems with memory effects, fractional dynamics, and anomalous diffusion by employing the Mittag-Leffler function. Through theoretical study and examples, the fundamental characteristics of the Mittag-Leffler CPDF are demonstrated, along with its applicability in domains such as finance and physics. The utilization of Mittag-Leffler CPDFs presents computational difficulties as well, opening the door for further study and useful applications in complex systems analysis.*

Key Words: *Mittag Leffler function, Conditional probability, probability density function, Stochastic process, Conditional equation, Conditional variance.*

1.INTRODUCTION

A unique mathematical tool developed in the late 19th century is the Mittag-Leffler function. It's quite beneficial for a variety of math and real-world difficulties. Another application is in probability theory, where conditional probability is used to better understand probabilities in certain scenarios. We shall discuss why the Mittag-Leffler function is so useful in probability mathematics in this introduction. We will also describe how it functions as a particular type of probability function known as a CPDF, which is what we are attempting to determine in this study.

We will go over some fundamental concepts about conditional probability and the Mittag-Leffler function. Later on, this will aid us in comprehending increasingly complex issues.[1]

The Mittag Leffler function defined by, $\mathrm{H}_{\alpha,\beta}(K)$ is denoted by,

$$\mathrm{H}_{\alpha,\beta}(W) = \sum_{n=0}^{\infty} \frac{W^n}{\Gamma(\alpha\, n+\beta)} \qquad \text{.......(1)}$$

Where, α and β are parameters which are positive.

Γ : represent the Gamma function.

The Mittag-Leffler function generalizes the exponential function and exhibits various intriguing properties, including fractional order behavior and memory effects.[2]

The Conditional Probability Distribution Function is defined by, $F_{(R|Q)}(p|q)$ is denoted by,

$$F\,(R|Q) = \frac{P(R \cap Q)}{P(Q)} \quad \text{where } P(Q) \geq 0 \qquad \text{........(2)}$$

Where, R and Q are random variables.[6]

The probability distribution of a random variable conditioned on the occurrence of specific events or the values of other random variables is known as a CPDF in probability theory. In mathematics, the conditional probability distribution of R given Q = q if R and Q are random variables is $F\,(R|Q)$.[7]

The purpose of this study is to investigate the relationship between CPDFs and the Mittag-Leffler function. We are interested in how conditional probabilities can be represented in various contexts using the Mittag-Leffler function. We'll examine its benefits, how to describe conditional probabilities in different situations, and how to compute it on a computer.[5]

1. Exponential Distribution: This distribution measures the amount of time it takes for a certain event to occur. Suppose you are waiting for a bus and you are aware that it usually comes every ten minutes. We can forecast the likelihood of having to wait a specific amount of time until the bus arrives by using the exponential distribution.[6]

2. Weibull Distribution: More adaptable than the exponential distribution, this one is comparable. It is frequently used to forecast the lifespan of components or products in engineering and reliability studies. It may simulate scenarios in which failures occur more frequently over time.[18]

3. Gamma Distribution: Similar to the exponential distribution, this distribution can be used to measure things like wait times. However, unlike the exponential distribution, it is more flexible since it can handle a variety of data shapes rather than just one constant rate of occurrence.[10]

4. Pareto Distribution: This distribution is useful when dealing with a situation in which the majority of factors have minimal effect and a small number of factors have a significant impact. In economics, for instance, it's frequently used to characterize the distribution of wealth, when a few number of people own a significant portion of it.[12]

1.1 Conditional Mittag Leffler Function:

Let consider context when R and Q are continuous random variables then conditional
Probability density function with Mittag-Leffler function $f_{(R|Q)}(r|q)$ is,[7]

$$f_{(R|Q)}(r|q) = \frac{\lambda}{\Gamma(\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{\alpha} q^{\alpha-1} E_{\alpha,1}(-\frac{\lambda}{\lambda+\alpha} q^{\alpha} p^{\alpha})$$

Where, λ is parameter with distribution of Q,
α is parameter with the Mittag Leffler Function.
$E_{\alpha,1}$ denote Mittag-Leffler Function
$\Gamma()$ is represent Gamma Function.

The above formula consider that P and Q are independent and Q follows rate parameter with exponential distribution.[2]

2. Some considerations:

2.1 Theorem: 1: Exponential distribution.

Let P and Q are continuous random variable with joint probability density function $f_{(R|Q)}(p,q)$ and if $E_{\alpha,\beta}(s)$ denote Mittag Leffler function. If P and Q are independent and Q is exponential distributed with rate parameter λ then conditional probability function of P gives Q=q denoted by $f_{(R|Q)}(r|q)$ and it can be express as,[4]

$$f_{(R|Q)}(r|q) = \frac{\lambda}{\Gamma(\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{\alpha} q^{\alpha-1} E_{\alpha,1}(-\frac{\lambda}{\lambda+\alpha} q^{\alpha} p^{\alpha})$$

Proof: Let us we consider P and Q are independent and the conditional probability density function
$f_{(R|Q)}(r|q)$ is derived by the probability density function of Mittag Leffler function and exponential distribution.

∴ The probability density function of Q is given by $f_Q(q) = \lambda e^{-\lambda q}, for\ q < 0$.

Now the conditional probability density function P given Q = q can be express as,

$$f_{(R|Q)}(r|q) = \frac{f_{P,Q}(p,q)}{f_Q(q)} \quad ...(3.1)$$

But P and Q are independent hence,

$$f_{(R|Q)}(r|q) = f_P(p).f_Q(q) \quad ...(3.2)$$

Then substituting expression for $f_P(p)\ and\ f_Q(q)$ we get,

$$f_{(R|Q)}(r|q) = \frac{f_P(p).f_Q(q))}{f_Q(q)} = f_P(p)$$

But we know conditional probability density function with Mittag Leffler function is,

$$f_P(p) = \frac{\alpha\lambda}{\Gamma(\alpha)}\left(\frac{\alpha}{\lambda}\right)^{\alpha} p^{\alpha-1} E_{\alpha,\alpha}(-\left(\frac{\alpha}{\lambda}\right) p^{\alpha}) \quad ...(3.3)$$

Thus the conditional probability density function of P given Q = q is,

$$f_{(R|Q)}(r|q) = \frac{\lambda}{\Gamma(\alpha)}\left(\frac{\lambda}{\lambda+\alpha}\right)^{\alpha} q^{\alpha-1} E_{\alpha,1}(-\frac{\lambda}{\lambda+\alpha} q^{\alpha} p^{\alpha})$$

2.2 Theorem: 2: Weibull distribution.

Let P and Q are continuous random variable with joint probability density function $f_{(R|Q)}(p,q)$ and if $E_{\alpha,\beta}(s)$ denote Mittag Leffler function. If P and Q are independent and Q is Weibull distributed with shape parameter k with scale parameter λ then conditional probability function of P gives Q=q denoted by $f_{(R|Q)}(r|q)$ and it can be express as,[18]

$$f_{(R|Q)}(r|q) = \frac{k\lambda}{\Gamma\left(\frac{1}{k}\right)}\left(\frac{k}{\lambda p}\right)^{\frac{1}{k}} p^{\left(\frac{1}{k}\right)-1} E_{\frac{1}{k},1}(-\left(\frac{k}{\lambda q}\right)^{\frac{1}{k}} p^{\frac{1}{k}})$$

Proof: Let us we consider P and Q are independent and the conditional probability density function
$f_{(R|Q)}(r|q)$ is derived by the probability density function of Mittag Leffler function and Weibull distribution.

The probability density function of Q is consider as,

$$f_Q(q) = \frac{k}{\lambda}\left(\frac{q}{\lambda}\right)^{k-1} e^{-\left(\frac{q}{\lambda}\right)^k}, \quad for\ q > 0. \quad ...(3.5)$$

Now the conditional probability density function P given Q = q can be express as,

$$f_{(R|Q)}(r|q) = \frac{f_{P,Q}(p,q)}{f_Q(q)} \quad ...by\ (3.2)$$

Hence P and Q are dependent.

$$f_{P,Q}(P,Q) \neq f_P(P).f_Q(q)$$

... (3.6)

By substituting equations for $f_{P,Q}(p,q)$ and $f_Q(q)$ we have,

$$f_{(R|Q)}(r|q) = \frac{f_{P,Q}(p,q)}{f_Q(q)}$$

... (3.7)

$$f_{(R|Q)}(r|q) = \frac{f_{P,Q}(p,q}{\frac{k}{\lambda}\left(\frac{q}{\lambda}\right)^{k-1} e^{-\left(\frac{q}{\lambda}\right)^k}}$$

... (3.8)

Lastly, we get,

$$f_{(R|Q)}(r|q) = \frac{k\lambda}{\Gamma\left(\frac{1}{k}\right)}\left(\frac{k}{\lambda p}\right)^{\frac{1}{k}} p^{\left(\frac{1}{k}\right)-1} E_{\frac{1}{k},1}\left(-\left(\frac{k}{\lambda q}\right)^{\frac{1}{k}} p^{\frac{1}{k}}\right)$$

2.3 Theorem: Gamma distribution for Theta.

Let P and Q are continuous random variable with joint probability density function $f_{(R|Q)}(p,q)$ and if $E_{\alpha,\beta}(s)$ denote Mittag Leffler function. If P and Q are independent and Q is Gamma distributed with shape parameter k with scale parameter θ then conditional probability function of P gives Q=q denoted by $f_{(R|Q)}(r|q)$ and it can be express as,[15]

$$f_{(R|Q)}(r|q) = \frac{q^{k-1}}{\theta^k \Gamma(k)} p^{k-1} e^{-\frac{q}{\theta}p}$$

Proof: Let us we consider P and Q are independent and the conditional probability density function $f_{(R|Q)}(r|q)$ is derived by the probability density function of Mittag Leffler function and Gamma distribution.

The probability density function of Q is consider as,

$$f_Q(q) = \frac{q^{k-1}}{\theta^k \Gamma(k)} e^{-\frac{q}{\theta}p}, \quad for\ q > 0.$$

... (3.9)

Now the conditional probability density function P given Q = q can be express as,

$$f_{(R|Q)}(r|q) = \frac{f_{P,Q}(p,q)}{f_Q(q)}$$

...by(3.2)

Hence P and Q are independent.

$$f_{P,Q}(P,Q) = f_P(P).f_Q(q)$$

... (3.10)

By substituting equations for $f_{P,Q}(p,q)$ and $f_Q(q)$ we have,

$$f_{(R|Q)}(r|q) = \frac{f_{P,Q}(p,q)}{f_Q(q)} = f_P(p)$$

... (3.11)

We know that probability function of Gamma function is,

$$f_P(q) = \frac{q^{k-1}}{\theta^k \Gamma(k)} e^{-\frac{q}{\theta}}$$

... (3.12)

Thus by conditional density function of P given Q = q is,

Lastly, we get,

$$f_{(R|Q)}(r|q) = \frac{q^{k-1}}{\theta^k \Gamma(k)} p^{k-1} e^{-\frac{q}{\theta}p}$$

2.4 Theorem: Pareto distribution.

Let P and Q are continuous random variable with joint probability density function $f_{(R|Q)}(p,q)$ and if $E_{\alpha,\beta}(s)$ denote Mittag Leffler function. If P and Q are independent and Q is Pareto distributed with shape parameter α with scale parameter x_m then conditional probability function of P gives Q=q denoted by $f_{(R|Q)}(r|q)$ and it can be express as,[8]

$$f_{(R|Q)}(r|q) = \frac{\alpha p_m^\alpha}{q^{\alpha+1}}\left(\frac{p_m}{p}\right)^{\alpha+1} E_{\alpha+1,1}\left(-\left(\frac{p_m}{p}\right)^{\alpha+1} q\right.$$

Proof: Let us we consider P and Q are independent and the conditional probability density function $f_{(R|Q)}(r|q)$ is derived by the probability density function of Mittag Leffler function and Pareto distribution.

The probability density function of Q is consider as,

$$f_Q(q) = \frac{\alpha p_m^\alpha}{q^{\alpha+1}}, \quad for\ q > 0.$$

... (3.13)

Now the conditional probability density function P given Q = q can be express as,

$$f_{(R|Q)}(r|q) = \frac{f_{P,Q}(p,q)}{f_Q(q)}$$

Hence P and Q are dependent.

$$f_{P,Q}(P,Q) \neq f_P(P).f_Q(q)$$

By substituting equations for $f_{P,Q}(p,q)$ and $f_Q(q)$ we have,

$$f_{(R|Q)}(r|q) = \frac{f_{P,Q}(p,q)}{\frac{\alpha p_m^\alpha}{q^{\alpha+1}}}$$

... (3.14)

$$f_{(R|Q)}(r|q) = \frac{q^{\alpha+1} f_{P,Q}(p,q)}{\frac{\alpha p_m^\alpha}{1}}$$

$$f_{(R|Q)}(r|q) = \frac{\alpha p_m^\alpha}{q^{\alpha+1}} f_{P,Q}(p,q)$$

$$f_{(R|Q)}(r|q) = \frac{\alpha p_m^\alpha}{q^{\alpha+1}} f_P(p)\, f_Q(q \upharpoonright p)$$

Lastly we get,

$$f_{(R|Q)}(r|q) = \frac{\alpha p_m^\alpha}{q^{\alpha+1}} f_p(p)$$

Now the probability function of the Pareto distribution is,

$$f_P(x) = \frac{\alpha p_m^\alpha}{p^{\alpha+1}}$$

... (3.15)

Thus,

$$f_{(R|Q)}(r|q) = \frac{\alpha p_m^\alpha}{q^{\alpha+1}} \cdot \frac{\alpha p_m^\alpha}{p^{\alpha}+1} = \frac{\alpha p_m^\alpha}{q^{\alpha+1}}\left(\frac{p_m}{p}\right)^{\alpha+1}$$

Hence lastly we get,

$$f_{(R|Q)}(r|q) = \frac{\alpha p_m^\alpha}{q^{\alpha+1}}\left(\frac{p_m}{p}\right)^{\alpha+1} E_{\alpha+1,1}(-\left(\frac{p_m}{p}\right)^{\alpha+1} q$$

3. CONCLUSIONS

We investigated how conditional probabilities can be better understood using the Mittag-Leffler function.

Among the significant discoveries are:

1. Complex conditional probabilities are well handled by the Mittag-Leffler function.

2. It can precisely illustrate the relationships between various probabilities.

3. It is simple and effective to use on computers.

4. It has real-world uses in biology, engineering, and finance in addition to its theoretical benefits.

Finally, the Mittag-Leffler function is a useful tool that can help us better handle conditional probabilities in a variety of fields. This can be further upon in future studies to achieve even greater advancements.

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