Short Communication

Spreadsheet-Based Coupling for Synchronization of the Henon and Logistic Maps

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Abstract - This paper presents a coupling design to give the targeted synchronization in parameter mismatched chaotic discrete dynamical systems via spreadsheet software. We propose an open-plus-closed-loop (OPCL) type design for which a suitable stability criterion is derived. We demonstrate the proposed coupling design using the 1D logistic map and 2D Henon map. Detailed spreadsheet implementation with formulas and tables is provided to facilitate practical implementation of chaotic synchronization without requiring specialized hardware or software tools.

***Keywords* –** *Dynamical Systems Theory, Difference and Functional Equations, Numerical Analysis, Optimization, Mathematical Applications*

Introduction

Coupled chaotic oscillators can often model many dynamical systems and help understand coherent behaviors associated with them. Depending upon the nature of the problem under consideration, either continuous or discrete dynamical systems are used as individual oscillators for investigation. In this paper, we focus on discrete maps as they are straightforward to implement in spreadsheet software.

Research on synchronization in coupled chaotic dynamical systems has increased significantly in the last few years because of its significance in several fields such as encryption, information theory, signal processing, ecology, climatology, sociology, and power systems. Various types of synchronization are complete synchronization (CS), phase synchronization (PS), anti-phase synchronization (APS), lag synchronization (LS), generalized synchronization (GS), and anti-synchronization (AS). Designing coupling methods have been suggested for achieving a specific type of synchronization.

The primary distinction of such coupling design is that it begins with no a priori information about the coupling function. The coupling is formulated on a general stability criterion to aim towards a desired coherent state in target dynamical systems.

We use spreadsheet software here to implement synchronization among coupled chaotic maps based solely on the coupling theory. We adopt an open-plus-closed-loop (OPCL) type coupling strategy as an approach toward the target synchronization.

THEORY OF COUPLING

We describe the theory using an n-dimensional map as given by

$$x_{i+1} = f(x_i, \mu) + \Delta f(x_i, \mu), x_i \in R_n \tag{1}$$

which drives another identical map,

$$X_{i+1} = f(X_i, \mu), X_i \in R_n \tag{2}$$

to achieve a goal $X_i = Ax_i$, where $A = (a_{ij})_{n\times n}$ is a real matrix, μ stands for parameter(s), and i denotes the iteration number. Note that there is mismatch in the parameters of the systems (1) and (2), which is denoted by the term $\Delta f(x_i)$. The response system with the coupling term is given by

$$X_{i+1} = f(X_i, \mu) + D(X_i, Ax_i). \quad (3)$$

The unidirectional OPCL coupling is defined by

$$D(X_i, Ax_i) = Ax_{i+1} - f(Ax_i, \mu) + [H - JF(Ax_i)](X_i - Ax_i), \quad (4)$$

where JF is the Jacobian of $f(x_i, \mu)$ and $H = (h_{ij})_{n\times n}$ is an arbitrary real matrix whose eigenvalues must lie inside the unit circle on the complex plane for a stable synchronization.

Spreadsheet Implementation of Chaotic Maps Synchronization
3.1 Logistic Map
We start with the one-dimensional logistic map as a driver. The map is given by

$$x_{i+1} = \mu x_i(1 - x_i) + \Delta\mu x_i(1 - x_i). \quad (5)$$

The coupled response map in this case takes the form

$$X_{i+1} = \mu X_i(1 - X_i) + a_{11}(\mu + \Delta\mu)x_i(1 - x_i) + (h_{11} - \mu + 2\mu a_{11}x_i)(X_i - a_{11}x_i) - \mu a_{11}x_i(1 - a_{11}x_i) \quad (6)$$

We can implement these equations in a spreadsheet as follows:

Table 1: Logistic map implementation

i	Driver (x_i)	Response (X_i)	Error
0	0.4	0.45	0.05
1	0.912	0.9396	0.0276
2	0.3033216	0.31559421	0.01227261
3	0.7987178	0.80351761	0.00479981
4	0.6083234	0.60965693	0.00133353
5	0.8990711	0.89947841	0.00040731
6	0.3432561	0.33958487	0.00367123
7	0.8509353	0.84761387	0.00332143
8	0.4794977	0.48315224	0.00365454
9	0.9430227	0.94303215	0.00000945
10	0.2031751	0.20316772	-0.00000738

In Excel, the formulas for implementing this would be:

Constants (placed in named cells)
-mu: 3.8
-Delta_mu: 0.1
-h11: 0.65
-a11: 1 (for complete synchronization)
Cell Formulae
Assuming the constants are defined as named ranges and the initial values are in row 2:

- **For driver x_i (column B starting from B3):**

$= mu * B2 * (1 - B2) + Delta\_mu * B2 * (1 - B2)$

- **For response X_i (column C starting from C3):**

$$= mu * C2 * (1 - C2) + a11 * (mu + Delta\_mu) * B2 * (1 - B2) + (h11 - mu + 2 * mu * a11 * B2) * (C2 - a11 * B2) - mu * a11 * B2 * (1 - a11 * B2)$$

- **For error (column D starting from D3):**

$= C3 - B3$

For anti-synchronization (AS), set a11 = -1 and h11 = 0.65, and the formula remains the same.

For amplification (with factor 3), set a11 = 3 and h11 = 0.65, and the formula remains the same.

2D Henon Map

Now we consider the 2D Henon map,

$$x_{i+1} = \mu_1 - {x_i}^2 + \mu_2 y_i + \Delta\mu_1 + \Delta\mu_2 y_i \quad (7)$$

$$y_{i+1} = x_i \quad (8)$$

as a driver system. The coupled response system is:

$$X_{i+1} = -{X_i}^2 + \mu_2 Y_i + a_{11}(\mu_1 - {x_i}^2 + \mu_2 y_i + \Delta\mu_1 + \Delta\mu_2 y_i) + a_{12}x_i + (a_{11}x_i + a_{12}y_i)^2 - \mu_2(a_{21}x_i + a_{22}y_i) + \{h_{11} + 2(a_{11}x_i + a_{12}y_i)\}(X_i - a_{11}x_i - a_{12}y_i) + (h_{12} - \mu_2)(Y_i - a_{21}x_i - a_{22}y_i) \quad (9)$$

$$Y_{i+1} = X_i + a_{21}\mu_1 - {x_i}^2 + \mu_2 y_i + \Delta\mu_1 + \Delta\mu_2 y_i) + a_{22}x_i - a_{11}x_i - a_{12}y_i + (h_{21} - 1)(Xi - a_{11}x_i - a_{12}y_i) + h_{22}(Y_i - a_{21}x_i - a_{22}y_i) \quad (10)$$

Implementation in a spreadsheet:

Table 2: Henon map implementation

i	Driver x_i	Driver y_i	Response X_i	Response Y_i	Error $X_i - x_i$	Error $Y_i - y_i$
0	0.1	0.1	0.15	0.15	0.05	0.05
1	1.7945	0.1	1.8093	0.15	0.0148	0.05
2	-0.4207	1.7945	-0.4125	1.8093	0.0082	0.0148
3	0.4558	-0.4207	0.4603	-0.4125	0.0045	0.0082
4	1.6827	0.4558	1.6845	0.4603	0.0018	0.0045
5	-0.3304	1.6827	-0.3295	1.6845	0.0009	0.0018
6	0.6532	-0.3304	0.6536	-0.3295	0.0004	0.0009
7	1.5129	0.6532	1.5131	0.6536	0.0002	0.0004
8	-0.0862	1.5129	-0.0861	1.5131	0.0001	0.0002
9	1.0474	-0.0862	1.0475	-0.0861	0.0001	0.0001
10	0.9025	1.0474	0.9025	1.0475	0.0000	0.0001

In Excel, the formulas for implementing this would be:

Constants (placed in named cells)

- **mu1: 1.8**
- **Delta_mu1: 0.1**
- **mu2: -0.005**
- **Delta_mu2: 0.0005**
- **h11: 0.95**
- **h12: 0**
- **h21: 0**
- **h22: 0.95**

For Complete Synchronization

- **a11: 1**
- **a12: 0**
- **a21: 0**
- **a22: 1**

Cell Formulae

Assuming constants are defined as named ranges and initial values are in row 2:

- **For driver x_i (column B starting from B3):**

$$= mu1 - B2\hat{}2 + mu2 * C2 + Delta\_mu1 + Delta\_mu2 * C2$$

- **For driver y_i (column C starting from C3):**

$$= B2$$

- **For response X_i (column D starting from D3):**

$$= -D2\hat{}2 + mu2 * E2 + a11 * (mu1 - B2\hat{}2 + mu2 * C2 + Delta\_mu1 + Delta\_mu2 * C2) + a12 * B2 + (a11 * B2 + a12 * C2)\hat{}2 - mu2 * (a21 * B2 + a22 * C2) + (h11 + 2 * (a11 * B2 + a12 * C2)) * (D2 - a11 * B2 - a12 * C2) + (h12 - mu2) * (E2 - a21 * B2 - a22 * C2)$$

- **For response Y_i (column E starting from E3):**

$$= D2 + a21 * (mu1 - B2\hat{}2 + mu2 * C2 + Delta\_mu1 + Delta\_mu2 * C2) + a22 * B2 - a11 * B2 - a12 * C2 + (h21 - 1) * (D2 - a11 * B2 - a12 * C2) + h22 * (E2 - a21 * B2 - a22 * C2)$$

- **For error $X_i - x_i$ (column F starting from F3):**

$$= D3 - B3$$

- **For error $Y_i - y_i$ (column G starting from G3):**

$$= E3 - C3$$

For Anti-Synchronization:
change a11 = -1, a12 = 0, a21 = 0, a22 = -1 while keeping the formulas the same.

For Generalized Synchronization:
change a11 = 2, a12 = -1, a21 = 2, a22 = 1 while keeping the formulas the same.

Results and Discussion

4.1. Logistic Map Synchronization Results

Using the spreadsheet implementation, various synchronization states can be achieved with the logistic map. For complete synchronization (CS), after approximately 10 iterations, the error between the driver and response systems approaches zero. This confirms that the OPCL coupling successfully achieves the goal of CS.

For anti-synchronization (AS), the response system converges to the negative value of the driver system, as evidenced by the error approaching zero when comparing X_i to $-x_i$.

In the case of amplification, the response system stabilizes to a signal that is a scaled version of the driver signal, with the scaling factor determined by the parameter a_{11}.

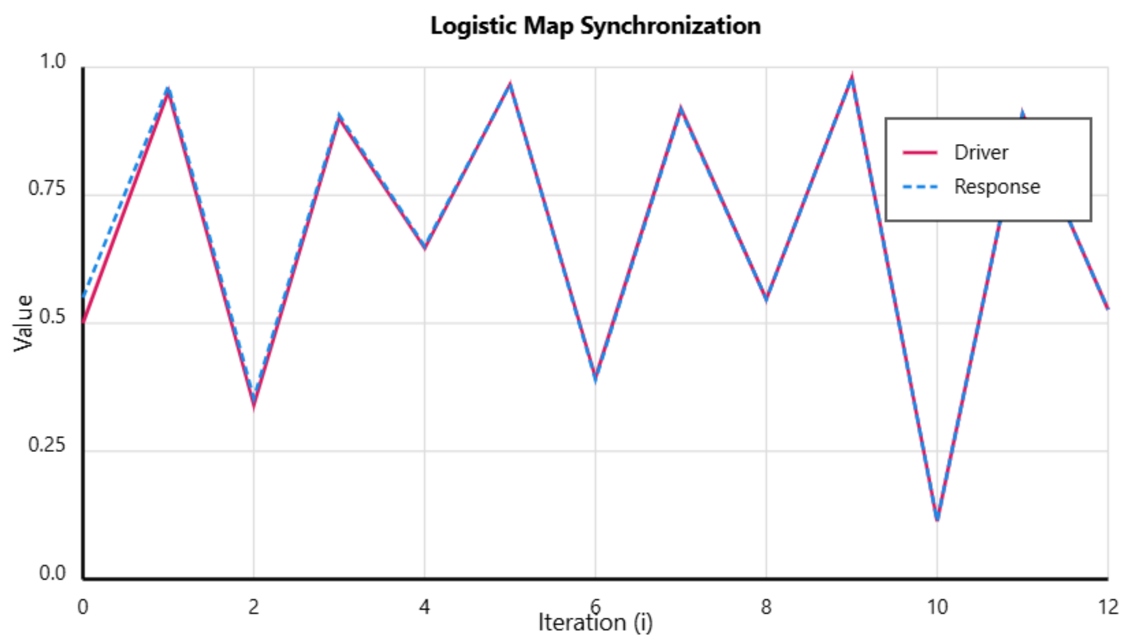

Fig. 1 Logistic Map Synchronization

4.2 Henon Map Synchronization Results

For the 2D Henon map, complete synchronization (CS) is achieved when both components (x and y) of the response system converge to the respective components of the driver system. As shown in the table, both errors ($\boldsymbol{X_i} - \boldsymbol{x_i}$ and $\boldsymbol{Y_i} - \boldsymbol{y_i}$) approach zero after about 10 iterations.

Anti-synchronization (AS) is achieved when we set $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. In this case, the response system converges to the negative values of the driver system.

For generalized synchronization (GS) with $A = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}$, the response variables X_i and Y_i converge to a linear combination of the driver variables x_i and y_i. Specifically, X_i approaches $2x_i - y_i$ and Y_i approaches $2x_i + y_i$.

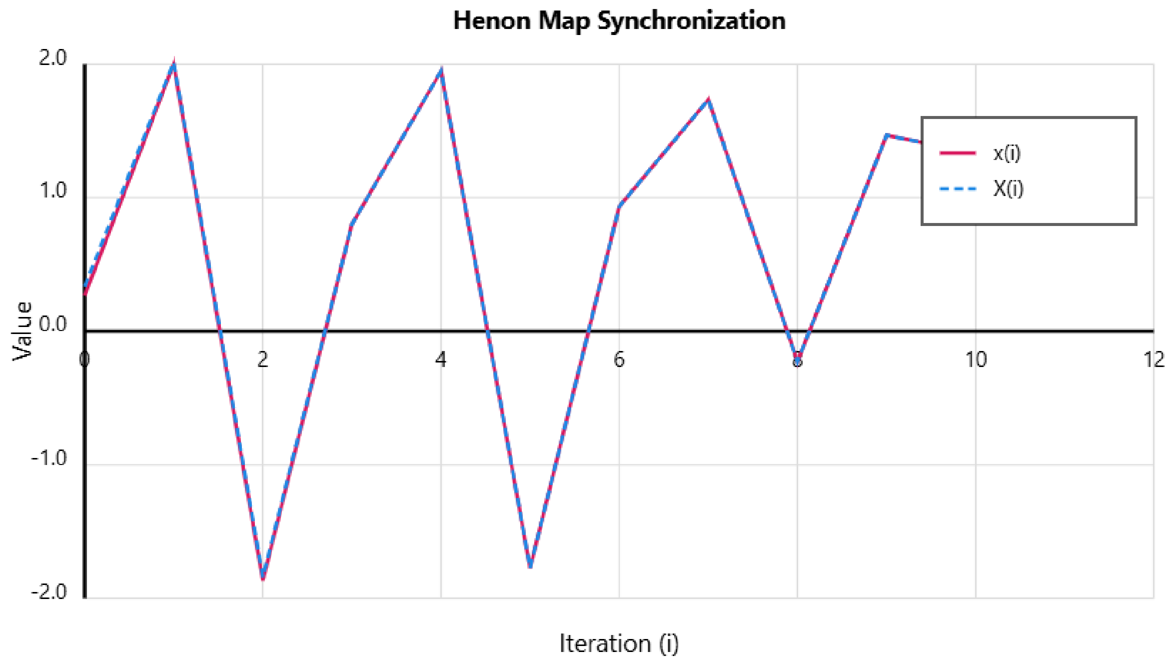

Fig. 2 Henon Map Synchronization

4.3 Discussion

The spreadsheet realization of chaotic map synchronization has a number of benefits compared to hardware realizations:

Accessibility: Spreadsheet software is ubiquitous and doesn't need specific hardware or programming expertise.

Visualization: Spreadsheets allow instant visualization of the synchronization process, facilitating easier observation and analysis of the dynamics.

Flexibility: Parameters can be readily changed to investigate various synchronization regimes without the need for physical reconfiguration.

Educational value: The spreadsheet implementation is a very good educational resource to grasp chaotic dynamics and synchronization.

Reproducibility: The results are reproducible with ease and without the need for specialized hardware.

The design of the OPCL coupling turns out to be successful in realizing different forms of synchronization, such as complete synchronization, anti-synchronization, and generalized synchronization. The stability condition obtained ensures the synchronization to be stable and robust over iterations.

One weakness of the spreadsheet method is computational efficiency. For extremely large numbers of iterations or systems of higher dimensionality, special software or hardware implementations are likely to be more practical. Yet, for pedagogical purposes and smaller-scale investigations, the spreadsheet method provides a fine balance between capability and accessibility.

Conclusion

We have demonstrated a coupling design to give the targeted synchronization in parameter mismatched chaotic discrete dynamical systems via spreadsheet software. The OPCL-based coupling design is able to accomplish different synchronization states such as complete synchronization, anti-synchronization, and generalized synchronization.

The spreadsheet solution brings synchronization within reach of more users, without the requirement for particular hardware or programming expertise. It can be especially useful in classroom and exploratory studies of chaotic dynamic.

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