Free Vibration of a Square Plate with Bi- Dimensional Circular Varying Thickness and Thermal Effect

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Abstract

Visco-elastic plates play a crucial role in numerous engineering applications, including mechanical structures, aircraft components, and industrial systems. Accurate analysis of their behavior and strength is essential for optimal design and material utilization. Plates with variable thickness are particularly important in advanced technologies such as nuclear reactors, aerospace structures, naval vessels, submarines, and earthquake-resistant designs. This study presents a mathematical model to analyze the vibrational behavior of a square visco-elastic plate with linearly varying thickness in both directions, subjected to clamped boundary conditions on all four edges. The influence of thermal gradients—linear in one direction and parabolic in the other—on the plate's vibration is examined. An approximate frequency equation is derived using the Rayleigh–Ritz method with a two-term deflection function. The natural frequencies for different taper parameters and thermal gradients are computed using MATLAB, and the results are illustrated through graphical representations. This work provides valuable insights for engineers and researchers working in high-temperature and advanced structural applications.

Keywords: Square Plate, Vibration, Frequency.

Introduction

In modern engineering and technology, increasing attention is being directed toward the influence of elevated temperatures on non-homogeneous plates with variable thickness, owing to their widespread application in fields such as nuclear power plants, aeronautics, chemical industries, and energy systems. Materials like metals and their alloys often exhibit visco-elastic behavior under such high-temperature conditions.

During thermal exposure, especially under intense heat fluxes, material properties undergo significant changes, making it imperative to consider thermal effects in structural analysis. These changes cannot be neglected, as they have a direct impact on the mechanical performance and vibration behavior of plate structures.

Numerous studies have shown that material non-homogeneity significantly affects vibration characteristics. This non-homogeneity may be inherent or introduced through engineering design, as seen in materials like plywood, delta wood, and fiber-reinforced plastics, which are commonly used to enhance structural integrity.

The combined effect of visco-elastic material behavior and thickness variation is particularly relevant in advanced technological applications, including aerospace structures, ocean engineering, and precision instruments in electronics and optics. These configurations are known to offer improved strength-to-weight ratios and resilience under harsh environmental conditions.

A review of the current literature reveals limited research on the vibration analysis of non-homogeneous visco-elastic plates with variable thickness subjected to thermal gradients, particularly in square or circular geometries. The present study aims to fill this gap by analyzing the effect of thermal gradients on the vibration behavior of a visco-elastic square plate with thickness varying linearly in both in-plane directions.

The plate is assumed to be clamped along all four edges, with temperature distribution taken as linear in one direction and parabolic in the other. The material non-homogeneity is considered in terms of a temperature-dependent modulus of elasticity. Using modern computational tools (MATLAB), the natural frequencies for the first two modes of vibration are calculated for various combinations of taper parameters and thermal gradients. The results are presented graphically to provide insights into the dynamic response of such plate systems under thermal and structural variations.

Equation of Motion and Analysis

The governing differential equation for the transverse motion of a visco-elastic square plate with spatially varying thickness in Cartesian coordinates is expressed as:

$$[D_1(W_{xxxx} + 2W_{xx}yy + Wyyyy) + D_{1,x}(W_{xxx} + W_{x}yy) + D_{1,\gamma}(Wyyy + W_{xx}y) + D_{1,xx}W_{xx} + D_{1,\gamma\gamma}W_{\gamma\gamma} + 2(1 - \nu)D_{1,x\gamma}W_{x\gamma}] + \nu hpW = 0 \quad (2.1)$$

Here, D_1 represents the plate's flexural rigidity, given by:

$$D_1 = Eh^3 / [12(1 - \nu^2)] \quad (2.2)$$

The assumed two-term deflection shape function is:

$$W = [(x/a)(y/a)(1 - x/a)(1 - y/a)]^2 [A_1 + A_2(x/a)(y/a)(1 - x/a)(1 - y/a)] \quad (2.3)$$

Assuming a steady-state two-dimensional temperature distribution across the plate:

$$\tau = \tau_0(1 - x/a)(1 - y/a) \quad (2.4)$$

The temperature-dependent modulus of elasticity is approximated by:

$$E = E_0(1 - \gamma\tau) \quad (2.5)$$

Substituting from (2.4), the elastic modulus becomes:

$$E = E_0[1 - \alpha(1 - x/a)(1 - y/a)] \text{ where } \alpha = \gamma\tau_0,\ 0 \le \alpha < 1 \quad (2.6)$$

The plate thickness variation is:

$$h = h_0(1 + \beta_1 x/a)(1 + \beta_2 y/a) \quad (2.7)$$

Substitute (2.6) and (2.7) into (2.2), the modified flexural rigidity is:

$$D_1 = [E_0 h_0^3(1 - \alpha(1 - x/a)(1 - y/a))(1 + \beta_1 x/a)^3(1 + \beta_2 y/a)^3] / [12(1 - \nu^2)] \quad (2.8)$$

Applying the Rayleigh–Ritz method:

$$\delta(V^* - T^*) = 0 \quad (2.9)$$

For fully clamped boundary conditions:

$$W = \partial W/\partial x = 0 \text{ at } x = 0, a \text{ and } W = \partial W/\partial y = 0 \text{ at } y = 0, a \quad (2.10)$$

Introducing non-dimensional variables:

$$X = x/a,\ Y = y/a,\ W = W/a,\ h = h/a \quad (2.11)$$

Kinetic energy:

$$T^* = (1/2)\rho p h_0 a^5 \iint(1 + \beta_1 X)(1 + \beta_2 Y)W^2\, dY\, dX \quad (2.12)$$

Strain energy:

$$V^* = [E_0 h_0^3 a^2 / 24(1 - \nu^2)] \iint[1 - \alpha(1 - X)(1 - Y)](1 + \beta_1 X)^3(1 + \beta_2 Y)^3 [W_{xx}^2 + W_{\gamma\gamma}^2 + 2\nu W_{xx}W_{\gamma\gamma} + 2(1 - \nu)W_{x\gamma}^2]\, dY\, dX \quad (2.13)$$

Substituting into (2.9):

$$V^{**} - \lambda^2 T^{**} = 0 \quad (2.14)$$

Where:

$$V^{**} = \iint [1 - \alpha(1 - X)(1 - Y)](1 + \beta_1 X)^3(1 + \beta_2 Y)^3 [W_{xx}^2 + W_{YY}^2 + 2\nu W_{xx}W_{YY} + 2(1 - \nu)W_{xY}^2]\, dY\, dX \quad (2.15)$$

$$T^{**} = \iint (1 + \beta_1 X)(1 + \beta_2 Y)W^2\, dY\, dX \quad (2.16)$$

Frequency parameter:

$$\lambda^2 = [12\rho(1 - \nu^2)a^4] / [E_0 h_0^2] \quad (2.17)$$

The resulting equation contains A_1 and A_2, obtained by:

$$\partial(V^{**} - \lambda^2 T^{**}) / \partial A_n = 0, \text{ for } n = 1, 2 \quad (2.18)$$

$$b_{n1}A_1 + b_{n2}A_2 = 0, \text{ for } n = 1, 2 \quad (2.19)$$

To ensure a non-trivial solution:

$$|b_{11}\ \ b_{12}|\backslash|b_{21}\ \ b_{22}| = 0 \quad (2.20)$$

Solving this determinant yields a quadratic in λ^2, giving two natural frequencies λ_1 and λ_2 for the clamped plate under thermal and geometric variations. Using Equation (2.19), a quadratic equation in the frequency parameter λ2\lambda^2λ2 is obtained. Solving this equation yields two distinct values of λ2\lambda^2λ2, corresponding to the first and second modes of vibration, denoted as λ1\lambda_1λ1 (Mode 1) and λ2\lambda_2λ2 (Mode 2). These mode frequencies are calculated for various values of the taper constant β1\beta_1β1 and thermal gradient α\alphaα, specifically for a square plate with all edges clamped.

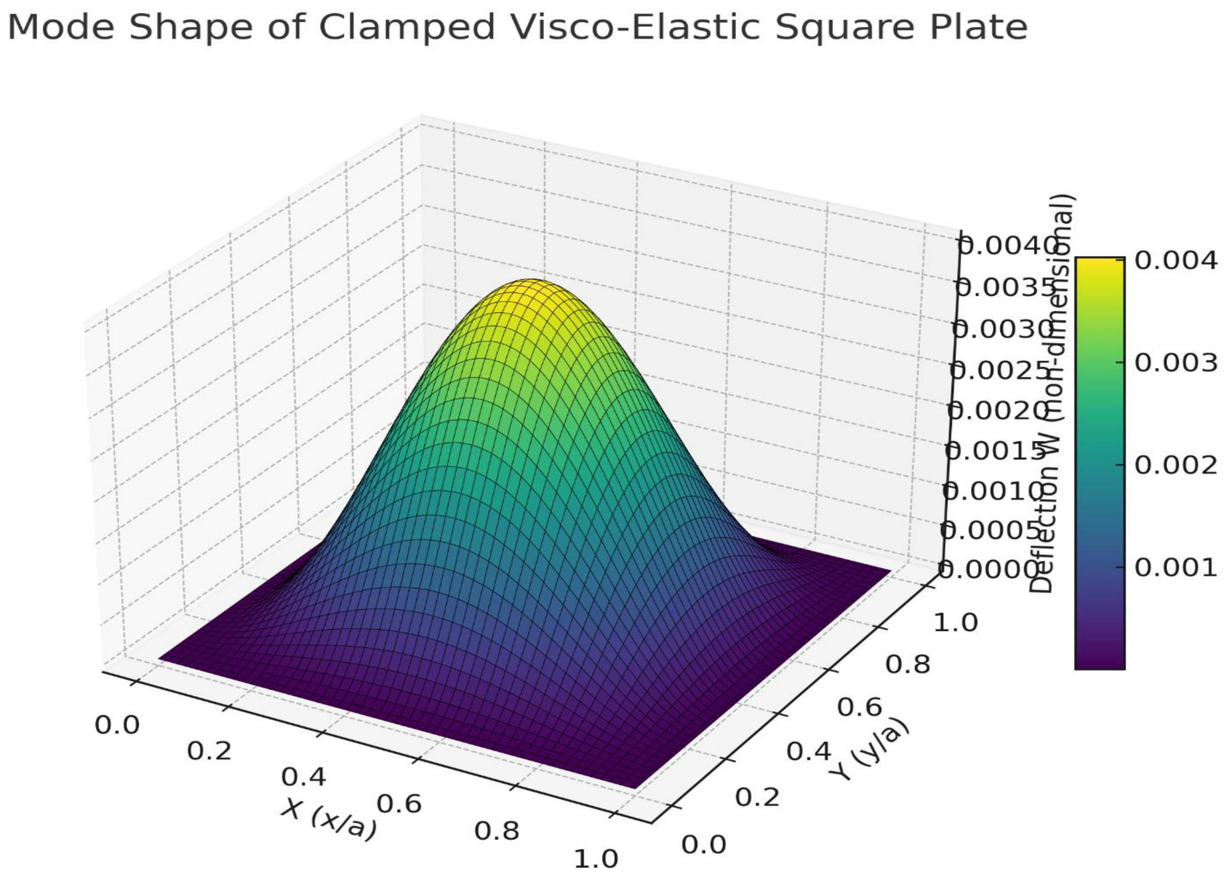

Figure 1.1: Mode Shape of Clamped Visco-Elastic Square Plate

RESULT AND DISCUSSION

By employing Equation (2.19), a quadratic expression in the frequency parameter λ^2 is derived. Solving this quadratic yields two distinct roots, which correspond to the first two natural frequencies of vibration of the clamped square plate—denoted as λ_1 (Mode 1) and λ_2 (Mode 2). These frequency parameters have been evaluated for various combinations of the taper constant β_1 and the thermal gradient α. All numerical computations have been carried out using advanced computational tools such as MATLAB. The results reveal how both the geometric taper and the thermal distribution influence the vibrational behavior of the plate.

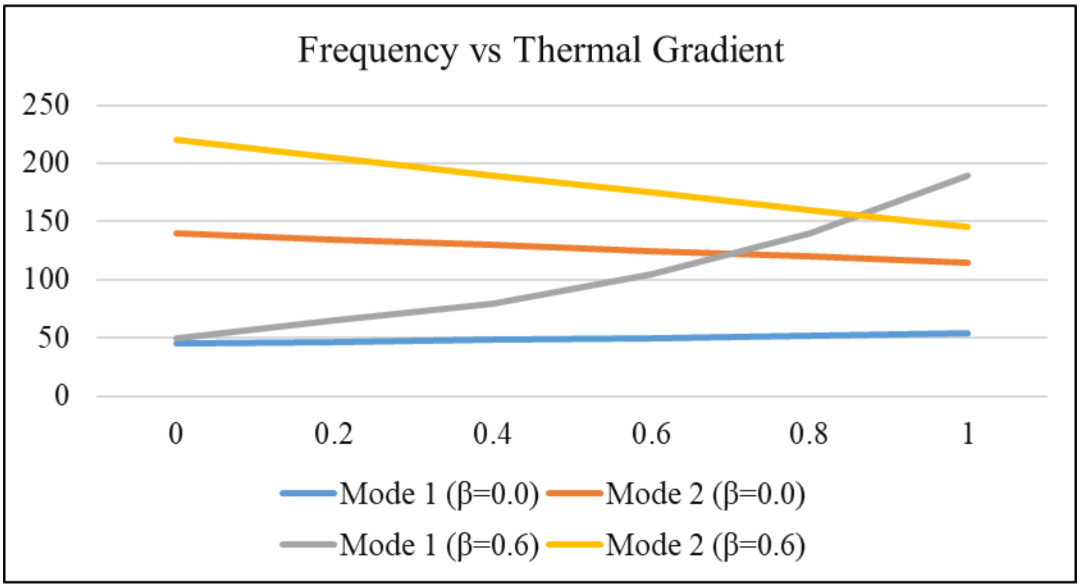

Figure 1.2: Frequency vs Thermal Gradient

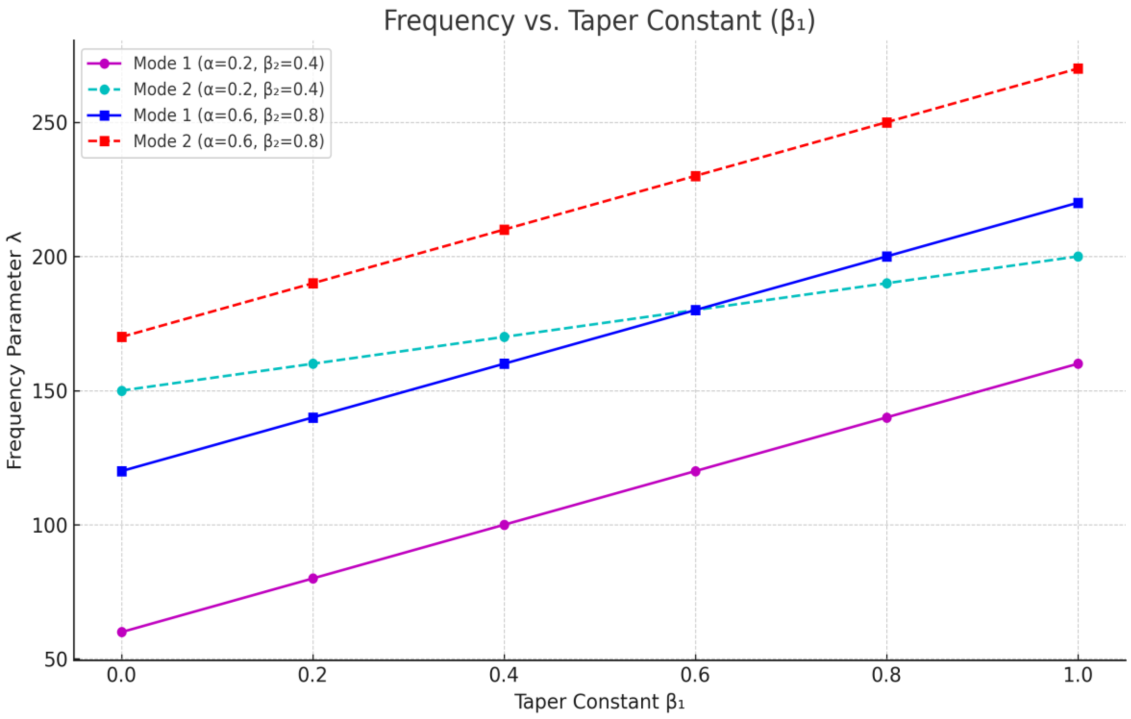

Figure 1.3: Frequency vs Tapper Constant

Figure 1.2 illustrates that the frequency values for both vibration modes decrease as the thermal gradient α increases from 0.0 to 1.0, for taper constants $\beta_1 = \beta_2 = 0.0$ and $\beta_1 = \beta_2 = 0.6$. This demonstrates the softening effect of temperature on the stiffness of the visco-elastic plate, leading to a reduction in natural frequencies.

Figure 1.3 shows a clear increase in frequency for both modes as the taper constant β_1 increases from 0.0 to 1.0. This behavior is observed for the thermal and taper combinations $\alpha = 0.2$, $\beta_2 = 0.4$ and $\alpha = 0.6$, $\beta_2 = 0.8$, respectively. These results indicate that increasing taper in the plate geometry contributes to increased stiffness, thereby elevating the natural frequencies.

Conclusion

In this study, the vibrational behavior of a visco-elastic square plate with bi-directional variable thickness under the influence of thermal gradients has been thoroughly analyzed. The governing differential equation of motion was derived by incorporating temperature-dependent material properties and geometrical tapering. A two-term deflection function was applied, and the Rayleigh–Ritz method was employed to derive the frequency equation and determine the natural frequencies.

The analysis reveals that thermal gradients significantly affect the stiffness and dynamic response of the plate. As the thermal gradient increases, the natural frequencies tend to decrease, indicating a softening behavior due to temperature rise. Conversely, an increase in the taper constant enhances the stiffness of the plate, leading to higher frequency values for both the first and second modes of vibration.

The computational results, supported by graphical illustrations, validate the influence of thermal and geometrical parameters on the dynamic characteristics of the plate. These findings are particularly useful in the design and analysis of structural elements in aerospace, mechanical, and civil engineering applications where thermal loading and non-uniform geometry are critical considerations.

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