Clique Transversal Domination of a Graph

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Abstract - ***Domination and partition are two interesting areas in graph theory. This paper introduces the concept of a clique transversal domination of a graph and studies some properties and bounds on its number. It is observed that this type of domination network is more reliable than other types of dominating networks. This variant is motivated by grouping a network into subnetworks such that any element of a subnetwork is equally important with others within the same subnetwork. There exists plenty of real life applications of the said variant in real life. In the areas of communication network and organizational set up the groups which are created on the basis of clique transversal domination can efficiently coordinate with each other and bring forth the desired outputs faster and earlier. In this paper, the clique transversal domination number has been studied for variety of standard graphs.***

Key Words: *Graphs, Clique, Partition, Domination.*

1. INTRODUCTION

Unless and otherwise specified, the graphs under consideration here are nontrivial, connected, simple, finite and undirected. For a graph G, V denotes its vertex set while E its edge set. Terms and notations used in this article may be found either in Harary [1] or in Narsingh Deo[2]. Unless stated, the graph G has n vertices and e edges. If $S \subseteq V$ then $\langle S \rangle$ is the subgraph of G induced by the vertices of S.

A graph in which every pair of vertices is adjacent is called a complete graph. A complete graph on n vertices is denoted by K_n. A clique of a graph G is a subgraph of G that is complete. Note that we do not require the clique to be maximal under inclusion. A set of vertices such that none of them are adjacent is known as an independent set. The maximum cardinality of an independent set is called independence number denoted by β. A graph is said to be null if all its vertices are isolated. Any shortest path between two vertices is often referred as a geodesic. The number of edges in any longest geodesic in a connected graph G is called its diameter denoted by $diam G$. The length of any longest cycle is known as circumference of a graph denoted by $c(G)$. A block is a nonseparable graph. A connected graph in which each block is complete is called a block graph.

A graph partition is related with its vertices or edges. More formally, a set of subset of vertices $\{V_1, V_2, \ldots, V_k\}$ is a vertex partition of the vertex set V if $V = \bigcup_{i=1}^{k} V_i$ and $V_i \bigcap_{i \neq j} V_j = \phi, \quad i, j = 1, 2, \ldots, k$. It is called a clique partition, if each induced subgraph $\langle V_i \rangle$ of this partition is a clique. Similarly, edge partition is defined. Here we consider the partition of vertices only. A set $S \subseteq V$ is called a transversal set with respect to some vertex partition $\{V_1, V_2, \ldots, V_k\}$ if S meets (has non-empty intersection with) all V_i, $i = 1, 2, \ldots, k$.

Domination is a widely applicable concept which we often find in numerous real life situations. Almost all the network problems involve the concept of domination. It arises in the context of facility location problems such as location of hospitals, bus-stands, offices, schools, stores, etc., It also arises in situations involving sets of representatives, communications, surveillance and so on.

The concept of domination exists in the world from the ancient times when people started solving puzzles such as a queen's problem (determining the minimum number of queens to be placed in a chess board such that all its squares are either attacked by a queen or occupied by a queen). But it was properly identified with its name and studied first by Ore [3]. A set $D \subseteq V$ is called a dominating set of G if every vertex in $V - D$ is adjacent to some vertex in D. A dominating set D with minimum number of vertices is called a minimum dominating set and its cardinality $|D|$ is the domination number denoted by $\gamma(G)$.

There are numerous research articles about domination and its applications. Domination is plenty in its varieties such as connected domination, independent domination, total domination, etc., and each has some specific application in real life. A comprehensive study on the fundamentals of domination is well presented in a book by Hayens et. al [4]. Partition related domination was first introduced by E.J. Cockayne and S.T. Hedetniemi, in [6] in which they introduced domatic partition of a graph, a partition of V into dominating sets.

The purpose of this article is to introduce the concept of clique transversal dominating sets in graphs. This variant is motivated by grouping a network into subnetworks such that any element of a subnetwork is equally important with others within the same subnetwork. Though plenty of applications of the said variant exist in real life, we list some of them.

In the context of a communication network, the cities of a zone need to be grouped in such a way that we can mount a facility in any city of a group so that it reaches other cities within the group and if there is any problem with any city offering the facility, one can shift the facility location immediately to any other city within the group so that there is no interruption with the network service.

In the context of an organizational set up, one needs to split its employees into teams so that various tasks of the company can be achieved easily. In such cases, teams can be made in which each member is equally important. It also provides the feature of appointing any member of the team as its leader (representative) so that if any leader quits the job or goes in a long leave, anyone in the team can be able to lead it successfully. Thus everyone can get an opportunity to lead the team.

2. CLIQUE TRANSVERSAL DOMINATING SET

2.1 Definition

Let G be any graph. A dominating set $D \subseteq V$ is called a clique transversal dominating set if D is transversal set of a minimum clique partition of V. The cardinality $|D|$ of such a minimum set D is called the clique transversal domination number of G denoted by $\gamma_{Kt}(G)$ and we refer any such minimum clique transversal dominating set as a γ_{Kt} – set.

3. MAIN RESULTS

3.1 Proposition

1. $\gamma_{Kt}(P_n) = \left\lceil \frac{n}{2} \right\rceil$
2. $\gamma_{Kt}(C_n) = \left\lceil \frac{n}{2} \right\rceil$
3. $\gamma_{Kt}(K_n) = 1$
4. $\gamma_{Kt}(K_{m,n}) = \max\{m, n\}$
5. $\gamma_{Kt}(Q_n) = 2^{n-1}$

The results are direct consequences of the definition of the clique transversal dominating set. Hence the proofs are omitted.

Clearly, every clique transversal dominating set is also a dominating set. Hence the following result is immediate.

3.2 Proposition

$$\gamma(G) \le \gamma_{Kt}(G)$$

The converse of the above proposition need not be true.

3.3 Proposition

Any graph possesses a clique transversal dominating set.

Proof. Any graph is either connected or disconnected. If G is a connected graph, one can easily partition its vertices into cliques. If G is a disconnected graph, then each of its components is a connected graph which in turn can be partitioned into cliques. The isolated vertices can be included as singleton sets in any such partition. Hence a clique transversal dominating set exists for any graph.

3.4 Proposition

Any set $D \subseteq V$ is a γ_{Kt} – set of a graph G if and only if, D contains exactly one member from each class of a minimum clique partition of G.

Proof. Let D be a γ_{Kt} – set of a graph G. Clearly, D is a dominating set of G and a transversal set of some minimum clique partition of the given graph G say, $\pi = \{V_1, V_2, \ldots, V_k\}$. Since each class of π is dominated by any of its vertices, by minimality, and transversal property, D has exactly one vertex of each class in π.
Conversely, let D be any set of distinct representatives of some minimum clique partition say, π of the given graph G. Clearly, D is a minimum transversal set of π and also a dominating set of G. Thus D is a γ_{Kt} – set of G.

3.5 Corollary

If we denote the order of a minimum clique partition of a graph G by $\theta(G)$ [5], $\gamma_{Kt}(G) = \theta(G)$.

Remark

In [5], Berge used the notation $\theta(G)$ to denote the minimum cardinality of a clique partition which is same as the parameter $\gamma_{Kt}(G)$. But the views and interests of Berge [5] and the author here on this parameter obviously differ in the sense that it is viewed in the dominance perspective here where as in [5] it is used to identify whether a graph is perfect or not.

3.6 Theorem

For any connected graph G, $1 \leq \gamma_{Kt}(G) \leq n-1$. The lower bound is attained if and only if G is complete while the upper bound is attained if and only if G is a star.

Proof. The lower bound of the above inequality trivially holds good for any connected graph G.

If G is a complete graph, then by itself it is a clique and hence no more clique partition is required for G. Thus $\gamma_{Kt}(G)=1$. Conversely, if $\gamma_{Kt}(G)=1$, the minimum order of a clique partition is one implying that G is complete.

Let us now establish the upper bound and its related part. Let G be a connected graph. Let v_1 and v_2 be any two adjacent vertices in G. Now $V=\{\langle v_1,v_2\rangle, v_3, v_4, \ldots, v_n\}$ is a clique partition (need not be minimum) and hence $\gamma_{Kt}(G) \leq n-1$.

Let G be a star graph. As the set of $n-1$ pendant vertices in G constitutes an independent set, no two of them can be present in a clique. We note that any vertex by itself is a clique. Hence these vertices form a minimum clique partition of size $n-1$. And remaining root vertex can be included with any of these cliques to yield another clique K_2. Thus the order of the minimum clique partition of G remains as $n-1$ resulting in $\gamma_{Kt}(G)=n-1$.

Conversely, let $\gamma_{Kt}(G)=n-1$. This implies that there exists a minimum clique partition say π of G of order $n-1$. By pigeonhole principle, we see that π includes: a pair and $n-2$ singleton sets. As π is a minimum clique partition, and no two independent vertices can lie in same clique, we note that there is an independent set say S of $n-1$ vertices. The remaining one more vertex say 'v' can no more be in the independent set S; also 'v' must be adjacent to all $n-1$ vertices of S; otherwise G would be disconnected. Thus G becomes a star graph and 'v' becomes its root vertex.

The result below on null graphs is straight forward and hence we omit its proof.

3.7 Theorem

If G is a null graph with n vertices, $\gamma_{Kt}(G)=n$

Every maximal independent set is a dominating set and hence if $\beta(G)$ denotes the cardinality of a biggest independent set of a graph G then, $\gamma(G) \leq \beta(G)$. No two independent vertices of a graph can lie in a clique and hence $\gamma_{Kt}(G) \geq \beta(G)$. Combining these inequalities, the following is evident.

3.8 Theorem

If $\beta(G)$ is the independence number of a graph G, $\gamma(G) \leq \beta(G) \leq \gamma_{Kt}(G)$.

Remark

The parameters $\gamma_{Kt}(G)$ and β need not be equal, though they may seem to be so. The following serves as an example for the same.

If G is an odd cycle of length $2k+1>3$ without chords (a chord is an edge between two non-consecutive vertices of a cycle), then $\gamma_{Kt}(G)=k+1$ and $\beta=k$

Any set of pendant vertices in any graph G of $n>2$ vertices is independent in nature. Hence the following inequality is inevitable.

3.9 Corollary

For any graph G with $n>2$ vertices and p pendant vertices $\gamma_{Kt}(G) \geq p$

Any graph G is said to be a chordal graph or a triangulated graph if each of its cycles of length four or more has a chord. A graph is said to be $\alpha - perfect$ [5], if for every subgraph induced by the subset of vertices of G, the independence number is equal to its minimum order of clique partition; in other words, $\beta = \gamma_{\mathrm{Kt}}$. Hence the following observation is immediate.

3.10 Theorem

If G is $\alpha - perfect$, $\gamma_{Kt}(G)=\beta$ where β is the independence number of G.

3.11 Theorem

If any graph G has a Hamiltonian path (or cycle), then $\gamma_{Kt}(G) \leq \left\lceil \frac{n}{2} \right\rceil$

Proof. If G is a graph with a Hamiltonian path or cycle, then all its vertices are included in it. Hence by proposition [3.1], $\gamma_{Kt}(G) \leq \left\lceil \frac{n}{2} \right\rceil$.

The following result though trivial may be useful in further investigations.

Observation

It is interesting to note that similar result does not exists in case of blocks, though any two vertices of a block lie on a common cycle [1]. The following is a counter example for the same.

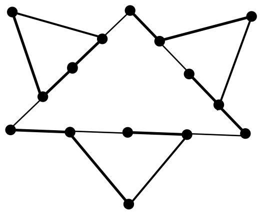

Figure -1: Graph *G*

In figure -1, G is a block with $n = 15$ and $\gamma_{Kt}(G) = 9$ which is not less than or equal to $\left\lceil \frac{n}{2} \right\rceil$.

3.12 Theorem

If the vertices of a graph G can be covered by k non-intersecting paths or cycles, $\gamma_{Kt}(G) \leq \left\lfloor \frac{n}{2} \right\rfloor + k$.

Proof. Let G be a graph satisfying the properties as given in the statement. Then the vertex set V of G can be partitioned as $V = \{V_1, V_2, \ldots, V_k\}$ such that each induced subgraph $\langle V_i \rangle$ has a Hamiltonian cycle or path. Obviously, $|V_1| + |V_2| + \ldots, + |V_k| = n$ and by theorem [3.7], $\gamma_{Kt}(\langle V_i \rangle) \leq \left\lceil \frac{|V_i|}{2} \right\rceil$.

This implies $\gamma_{Kt}(G) \leq \sum_{i=1}^{k} \left\lceil \frac{|V_i|}{2} \right\rceil \leq \left\lfloor \frac{n}{2} \right\rfloor + k$

3.13 Theorem

If a set of nonintersecting triangles of a graph G cover the vertices of G, $\gamma_{Kt}(G) \leq \frac{n}{3}$.

Proof. Each triangle is a clique that includes three vertices of the graph. As each is vertex-disjoint, the set of vertices of all such triangles form a clique partition of V and a member from each of them constitute a clique transversal dominating set which deduces the result.

3.14 Theorem

If the removal of pendant vertices of a graph yields either a cycle or path, then $\gamma_{Kt}(G) \leq \left\lfloor \frac{n+p}{2} \right\rfloor$ where p is the number of pendant vertices of G.

Proof. Let G be a connected graph; P be its set of p pendant vertices. Let us construct a clique partition π of V as follows. Remove the set P of p pendant vertices of G and include them as singleton sets (K_1) in π. In the induced subgraph $\langle G - P \rangle$, remove all the vertex-disjoint edges (K_2) and include them in π. Clearly, there may be at most one vertex say 'v' left behind and it can be so chosen such that it's a neighbour to some pendent vertex which is included as a K_1-clique in π. Combining these two vertices yields a K_2-clique in π. Now π is a clique partition of G with cardinality $p + \left\lfloor \frac{n-p}{2} \right\rfloor = \left\lfloor \frac{n+p}{2} \right\rfloor$. The graph G may have cliques of size three or more in which case this cardinality can be further reduced. Any transversal set S comprising exactly one member from each clique in π is a clique transversal dominating set implying that $\gamma_{Kt}(G) \leq \left\lfloor \frac{n+p}{2} \right\rfloor$.

3.15 Theorem

In any connected graph G, $\gamma_{Kt}(G) \leq n - \left\lceil \frac{diam\, G}{2} \right\rceil$.

Proof. Any longest geodesic in a connected graph covers $diam\, G + 1$ vertices and it has $\left\lceil \frac{diam\, G}{2} \right\rceil$ vertex-disjoint edges which cover all or all but one vertex of the longest geodesic. These disjoint-edges and a single vertex (if any) are cliques and hence they form a clique partition for the geodesic depending on odd or even nature of its number of vertices on this geodesic. These edges and the left out vertex are obviously cliques and their cardinality is

$\left\lceil \frac{diam\,G+1}{2} \right\rceil$. In the remaining graph as each vertex is a clique we have,

$\gamma_{Kt}(G) \le n-(diam\,G+1)+\left\lceil \frac{diam\,G+1}{2} \right\rceil \le n-\left\lceil \frac{diam\,G}{2} \right\rceil$.

We know that $c(G)$ denotes the circumference of a graph G. There are $c(G)$ vertices along this circumference which have a clique transversal partition of size $\left\lceil \frac{c(G)}{2} \right\rceil$ and obviously the remaining $\left\lfloor \frac{c(G)}{2} \right\rfloor$ vertices can be left out from a clique transversal dominating set which results in the following.

3.16 Theorem

In any connected graph G, $\gamma_{Kt}(G) \le n-\left\lfloor \frac{c(G)}{2} \right\rfloor$, where $c(G)$ is the circumference of G.

3.17 Theorem

If G is a block graph with b blocks, $\gamma_{Kt}(G)=b$.

Proof. Every block in a block graph G is a clique. Hence vertices of each block constitute a subset in the clique partition of V. Any set of distinct representatives of this partition obviously form a clique transversal dominating set. Hence the result follows.

3.18 Theorem

Let ω be the size of a largest clique in G, then $\gamma_{Kt}(G) \ge \left\lceil \frac{n}{\omega} \right\rceil$.

Proof. There exists no clique of size more than ω in any graph G. Thus any clique partition of V can include at most ω vertices in a clique from which we observe the above fact.

Any clique size cannot exceed the maximum degree plus one. If we denote the maximum degree of the graph by the symbol Δ, the following result is a direct outcome of the above theorem.

3.19 Corollary

If G is any graph, $\gamma_{Kt}(G) \ge \left\lceil \frac{n}{\Delta+1} \right\rceil$.

So far, all the results are relating the invariant $\gamma_{Kt}(G)$ with the number of vertices of the graph. Following is a result which gives some idea about the number of edges and the parameter $\gamma_{Kt}(G)$.

3.20 Theorem

If every set in a minimum clique transversal partition of a connected graph G, has k or more vertices,

$\gamma_{Kt}(G) \le \left\lceil \frac{e+1}{{}^kC_2+1} \right\rceil$ where e is the number of edges in G.

Proof. Let π be a minimum clique partition of the vertex set V that corresponds to the invariant $\gamma_{Kt}(G)$. We know that a clique of size k has $\frac{k(k-1)}{2}$ or ${}^kC_2$ edges. We infer the following (1) Each induced subgraph $\langle V_i \rangle$ is a clique where V_i is an element in π. (2) The cardinality of each V_i is k or more. (3) the number of edges in each induced subgraph $\langle V_i \rangle$ is ${}^kC_2$ or more. (4) There are γ_{Kt} induced cliques in π. Hence the total number of edges from of all the induced cliques of π is ${}^kC_2\gamma_{Kt}$ or more. Since G is a connected graph the γ_{Kt} cliques must be joined by $\gamma_{Kt}-1$ or more edges. Thus if e is the number of edges of G, then

$e \ge {}^kC_2\gamma_{Kt}+\gamma_{Kt}-1 \Rightarrow e+1 \ge \gamma_{Kt}({}^kC_2+1) \Rightarrow \gamma_{Kt} \le \frac{e+1}{{}^kC_2+1}$.

4. CONCLUSION

In this article the author has introduced a new type of graph theory related variant namely clique transversal domination in graphs which can be used to build more stronger teams and networks in social and communication networks. The variant has been studied under for variety of graphs and its bounds are determined. As a future work the readers can extend the results and study the applications of the parameters in a wider sense.

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