In insight into $QAC_2^{(1)}$: Dynkin diagrams and properties of roots

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Abstract - ***In this paper the indefinite quasi affine Kac Moody algebras $QAC_2^{(1)}$ are considered; basic properties of real and imaginary roots are studied for three specific classes of quasi affine Kac Moody algebras belonging to the family $QAC_2^{(1)}$; the short and long real roots are identified; the minimal imaginary roots and isotropic roots up to height 3 have been computed for these classes of Kac Moody algebras; These algebras also satisfy the purely imaginary property i.e. all the imaginary roots are purely imaginary ; All the 909 non isomorphic connected Dynkin diagrams associated with $QAC_2^{(1)}$, are given in the classification theorem. It is also proved that there are 648 Dynkin diagrams of extended hyperbolic type, 8 hyperbolic type and 253 diagrams of indefinite, non extended hyperbolic type in the classification of Dynkin diagrams associated with*** $QAC_2^{(1)}$.

Key Words: *Kac Moody algebras, Dynkin diagrams, roots, real , imaginary,, quasi hyperbolic, quasi affine.*

AMS Classification :17B67

1. INTRODUCTION

The subject Kac Moody algebras was developed by Kac and Moody in 1968, simultaneously and independently ([5],[9]). Among the three main types of Kac Moody algebras, namely finite, affine and indefinite, the study of indefinite type of Kac Moody algebras is still wide open, whose structure is not completely understood; The Kac Moody algebras were completely classified through Dynkin digrams for the finite, affine and, hyperbolic types ([5], [25]). The root systems of the finite and affine types have been characterized ([5],[9],[25]); As far as the imaginary roots of the indefinite type Kac Moody algebras are concerned, strictly imaginary roots were studied by Casperson [3] and special imaginary roots by Bennett [2]; Purely imaginary roots were introduced by Sthanumoorthy and Uma Maheswari ([10] ,[11]) ; Among the indefinite Kac Moody algebras, hyperbolic Kac Moody algebras , which are natural extensions of affine type were studied by Feingold & Frenkel [4], Benkart , Kang & Misra ([1],); Kang studied the structure of $HA_1^{(1)}$, $HA_2^{(1)}$ and $HA_n^{(1)}$ ([6]-[8]);Other sub classes of the indefinite type, namely Extended hyperbolic were introduced by Sthanumoorthy & Uma Maheswari ([10],[11]) and determined the structure and computed root multiplicities up to level 5 for $EHA_1^{(1)}$, $EHA_2^{(2)}$([12]-[14]) ;

Quasi hyperbolic and Quasi affine types were introduced by Uma Maheswari ([15],[18]). Uma Maheswari and et.al studied Quasi hyperbolic algebras QHG2,$QHA_2^{(1)}$, $QHA_4^{(2)}$, $QHA_7^{(2)}$, $QHA_5^{(2)}$ ([20]-[24]), quasi hyperbolic Dynkin diagrams of rank 3 in [16] and quasi affine Kac Moody algebras $QAD_3^{(2)}$, $QAG_2^{(1)}$ ([18]-[19]). In [17], the family $QAC_2^{(1)}$ was defined, the structure of a specific class of $QAC_2^{(1)}$ was determined using homological techniques and the general form of the Dynkin diagrams associated with this family $QAC_2^{(1)}$ was given;

Here in this paper, the complete classification of non isomorphic connected Dynkin diagrams (909 in number) associated with $QAC_2^{(1)}$ is given; Also, we consider three particular classes belonging to $QAC_2^{(1)}$ and study the basic properties of real and imaginary roots for the associated Quasi affine Kac Moody algebras $QAC_2^{(1)}$.

2. PRELIMINARIES

DEFINITION 1: A realization of a matrix $A=(a_{ij})_{i,j=1}^{n}$ is a triple $(H,\Pi,\Pi^{\vee})$ where l is the rank of A, H is a 2n - l dimensional complex vector space, $\Pi=\{\alpha_1,\alpha_2,...,\alpha_n\}$ and $\Pi^{\vee}=\{\alpha_1^{\vee},\alpha_2^{\vee},...,\alpha_n^{\vee}\}$ are linearly independent subsets of H* and H respectively, satisfying $\alpha_j(\alpha_i^{\vee})=a_{ij}$ for i, j = 1,….,n. Π is called the root basis. Elements of Π are called simple roots. The root lattice generated by Π is $Q=\sum_{i=1}^{n} z\alpha_i$

The Kac-Moody algebra g(A) associated with a GCM $A=(a_{ij})_{i,j=1}^{n}$ is the Lie algebra generated by the elements ei , fi , i=1,2,...n and H with the following defining relations :

$$[h,h'] = 0, \quad h,h' \in H$$
$$[e_i, f_j] = \delta_{ij}\alpha_i^{\vee}$$
$$[h,e_j] = \alpha_j(h)e_j$$
$$[h, f_j] = -\alpha_j(h) f_j, \quad i, j \in N$$
$$(ade_i)^{1-a_{ij}} e_j = 0$$
$$(adf_i)^{1-a_{ij}} f_j = 0, \forall\ i \neq j,\ i, j \in N$$

The Kac-Moody algebra g(A) has the root space decomposition $g(A) = \underset{\alpha\in Q}{\oplus} g_\alpha(A)$ where $g_\alpha(A) = \{x \in g(A)/[h,x] = \alpha(h)x, \ for\ all\ h \in H\}$.

An element α, $\alpha \neq 0$ in Q is called a root if $g_\alpha \neq 0$.

For any $\alpha \in Q\ and\ \alpha = \sum_{k=1}^{n} k_i\alpha_i$ define support of α, written as supp α, by supp $\alpha = \{i \in N / k_i \neq 0\}$.

Let $\Delta = (=\Delta(A))$ denote the set of all roots of g(A) and Δ_+ the set of all positive roots of g(A);

$\Delta_- = -\Delta_+$ and $\Delta = \Delta_+ \cup \Delta_-$.

Definition 2:[4]A GCM A is called symmetrizable if DA is symmetric for some diagonal matrix D = diag(q1,...,qn), with qi> 0 and qi's are rational numbers.

Note that , a GCM $A=(a_{ij})_{i,j=1}^{n}$ is symmetrizable if and only if there exists an invariant, bilinear, symmetric, non-degenerate form on g(A).

Definition 3: [4][25] To every GCM A is associated a Dynkin diagram S(A) defined as follows: (A) has n vertices and vertices I and j are connected by max{$|a_{ij}|$, $|a_{ji}|$} number of lines if $a_{ij}.a_{ji} \leq 4$ and there ia an arrow pointing towards I if $|a_{ij}|>1$. If $a_{ij}.a_{ji}>4$, i and j are connected by a bold faced edge, equipped with the ordered pair ($|a_{ij}|$, $|a_{ji}|$) of integers.

Definition 4: [17] Let $A=(a_{ij})_{i,j=1}^{n}$, be an indecomposable GCM of indefinite type. We define the associated Dynkin diagram S(A) to be of Quasi Affine (QA) type if S(A) has a proper connected sub diagram of affine type with n-1 vertices. The GCM A is of QA type if S(A) is of QA type. We then say the Kac-Moody algebra g(A) is of QA type.

Definition 5[9]:A root $\alpha \in \Delta$ is called real, if there exists a $w \in W$ such that w(α) is a simple root, and a root which is not real is called an imaginary root. An imaginary root α is called isotropic if (α,α) = 0.

$\alpha \in \Delta_+^{im}$ is called a minimal imaginary root (MI root, for short) if α is minimal in Δ_+^{im} with respect to the partial order on H*. By the symmetry of the root system, it suffices to prove the results for positive imaginary roots only.

Definition 6[3]: A root $\gamma \in \Delta^{im}$ is said to be strictly imaginary if for every $\alpha \in \Delta^{re}$, $\alpha+\gamma\ or\ \alpha-\gamma$ is a root. The set of all strictly imaginary roots is denoted by Δ^{sim}.

Definition 7[15]: A root $\alpha \in \Delta_+^{im}$ is called purely imaginary if for any $\beta \in \Delta_+^{im}$, $\alpha+\beta \in \Delta_+^{im}$.The Kac – Moody algebra is said to have the purely imaginary property if every imaginary root is purely imaginary.

3. CLASSIFICATION OF DYNKIN DIAGRAMS ASSOCIATED WITH $QAC_2^{(1)}$

In this section we explicitly give all the 909 non isomorphic connected Dynkin diagrams associated with the indefinite Quasi affine Kac-Moody algebras $QAC_2^{(1)}$.

Also we identify the extended hyperbolic and indefinite non extended hyperbolic diagrams in this family. It is interesting to note that this family OF Kac Moody algebras satisfy the purely imaginary property.

The general representation of the GCM associated with $QAC_2^{(1)}$ is $A = \begin{pmatrix} 2 & -a & -b & -c \\ -l & 2 & -1 & 0 \\ -m & -2 & 2 & -2 \\ -n & 0 & -1 & 2 \end{pmatrix}$, a,b,c,l,m,n are non negative integers not all 0 simultaneously;

Note that A is symmetrizable if 2bn=cm and am=2bl.

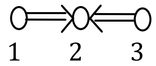
i.e. A = DB, where

$$D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & l/a & 0 & 0 \\ 0 & 0 & m/b & 0 \\ 0 & 0 & 0 & n/c \end{pmatrix},\ B = \begin{pmatrix} 2 & -a & b & -c \\ -a & 2a/l & -a/l & 0 \\ -b & -a/l & 2b/m & -2b/m \\ -c & 0 & -2b/m & 2c/n \end{pmatrix}.$$

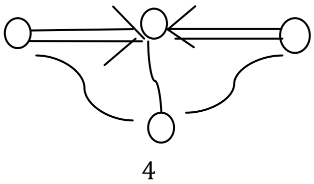
It can also be represented as $\begin{pmatrix} 2 & -1 & 0 & -a \\ -2 & 2 & -2 & -b \\ 0 & -1 & 2 & -c \\ -l & -m & -n & 2 \end{pmatrix}$.

Theorem 1 (ClassificationTheorem):. There are 909 non isomorphic, connected Dynkin diagrams associated with the indefinite Quasi affine Kac-Moody algebra $QAC_2^{(1)}$

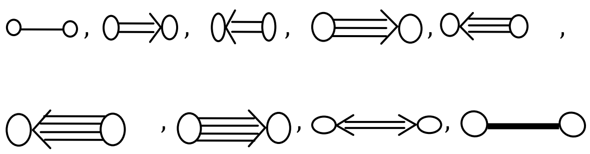
Proof. Consider the affine Kac-Moody algebra $C_2^{(1)}$ with 3 vertices whose Dynkin diagram is given by



Add the fourth vertex, which is connected to atleast one of the three vertices of $C_2^{(1)}$.

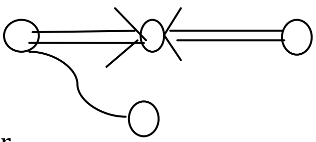


Here  denotes the 9 possible connections of vertex 4 with the other 3 vertices.



The fourth vertex can be connected to the existing three vertices by one the following cases:

Case i) Fourth vertex is connected to exactly one of the three vertices1,2,3. In this case, fourth vertex can be connected to the i^{th} vertex, (i=1,2,3) by the 9 possible edges given above.



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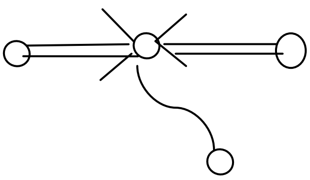
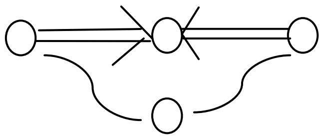


Thus, possible number connected Dynkin diagrams associated with $QAC_2^{(1)}$ in this case is 2 x 9 = 18 (Note that, we are considering non isomorphic diagrams; Joining the 4th vertex with either 1st or 3rd vertices will result in isomorphic diagrams)

Case ii) Fourth vertex is connected to exactly two of the 3 vertices; To get non isomorphic diagrams, these two vertices can be chosen from the three vertices in 2 possible ways. (i.e.(1,4) and (2,4) can be joined or (1,4) and (3,4) can be joined.)



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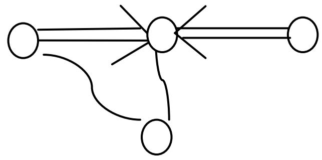
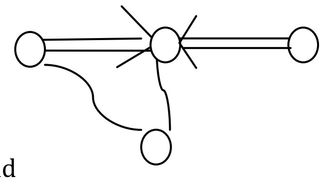


Note that joining (2,4) and (3,4) will result in a isomorphic set of diagrams with (1,4), (2,4) connections . i.e. the two Dynkin diagrams



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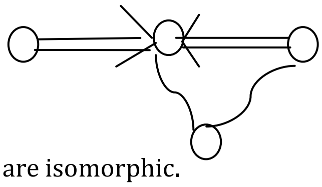
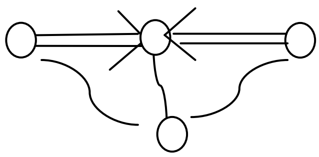


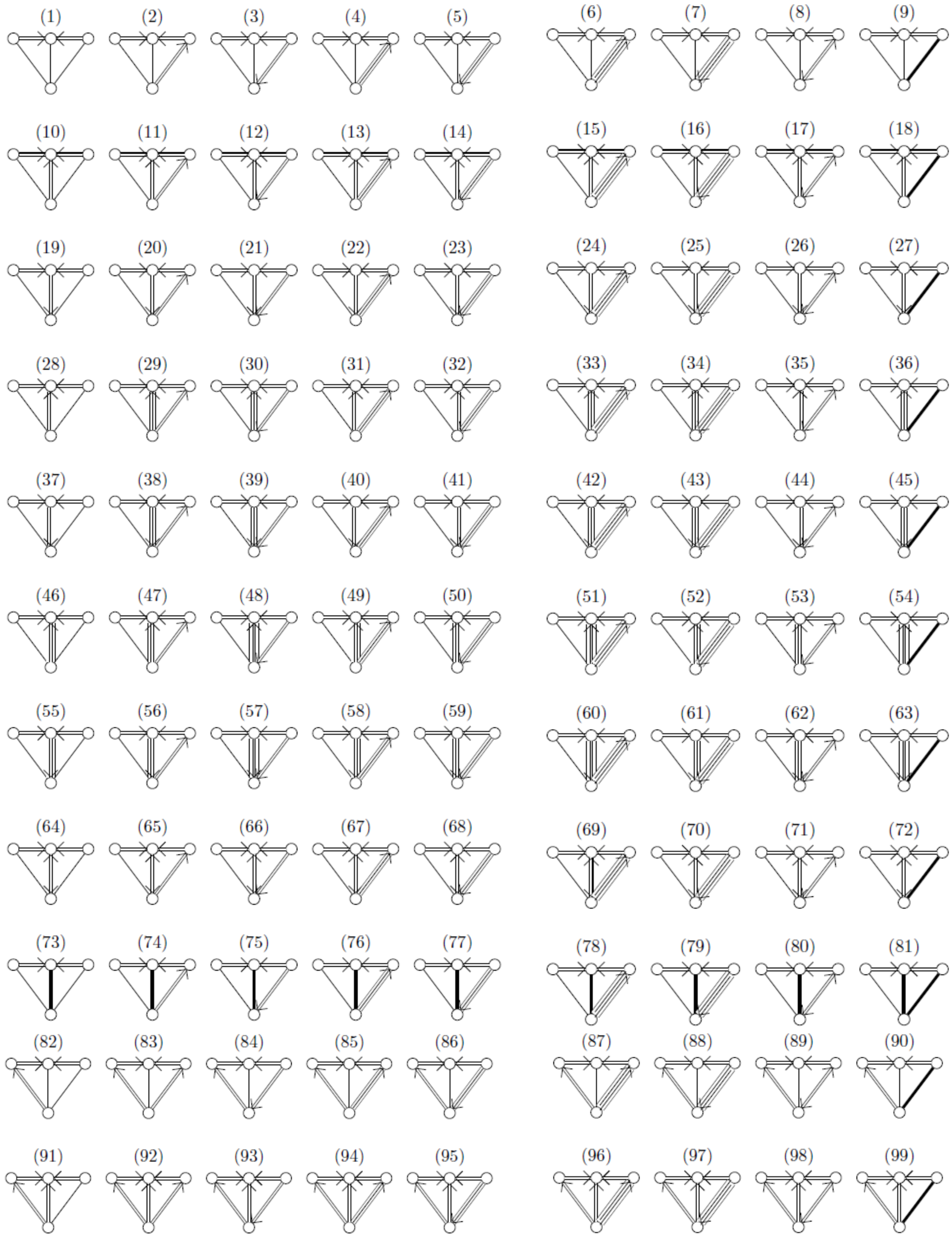
are isomorphic.

For each possibility, the remaining edge can be connected by any of the above mentioned 9 edges.
Hence. in this case, the associated connected Dynkin diagrams are $2x\ 9^2 = 162$

Case iii) Fourth vertex is connected to all the 3 vertices. In this case, there are 9^3 connected Dynkin diagrams.



Totally there are 18 + 162 + 729 =909 non isomorphic, connected Dynkin diagrams associated with $QAC_2^{(1)}$.
All these 909 Dynkin diagrams are drawn in the following table.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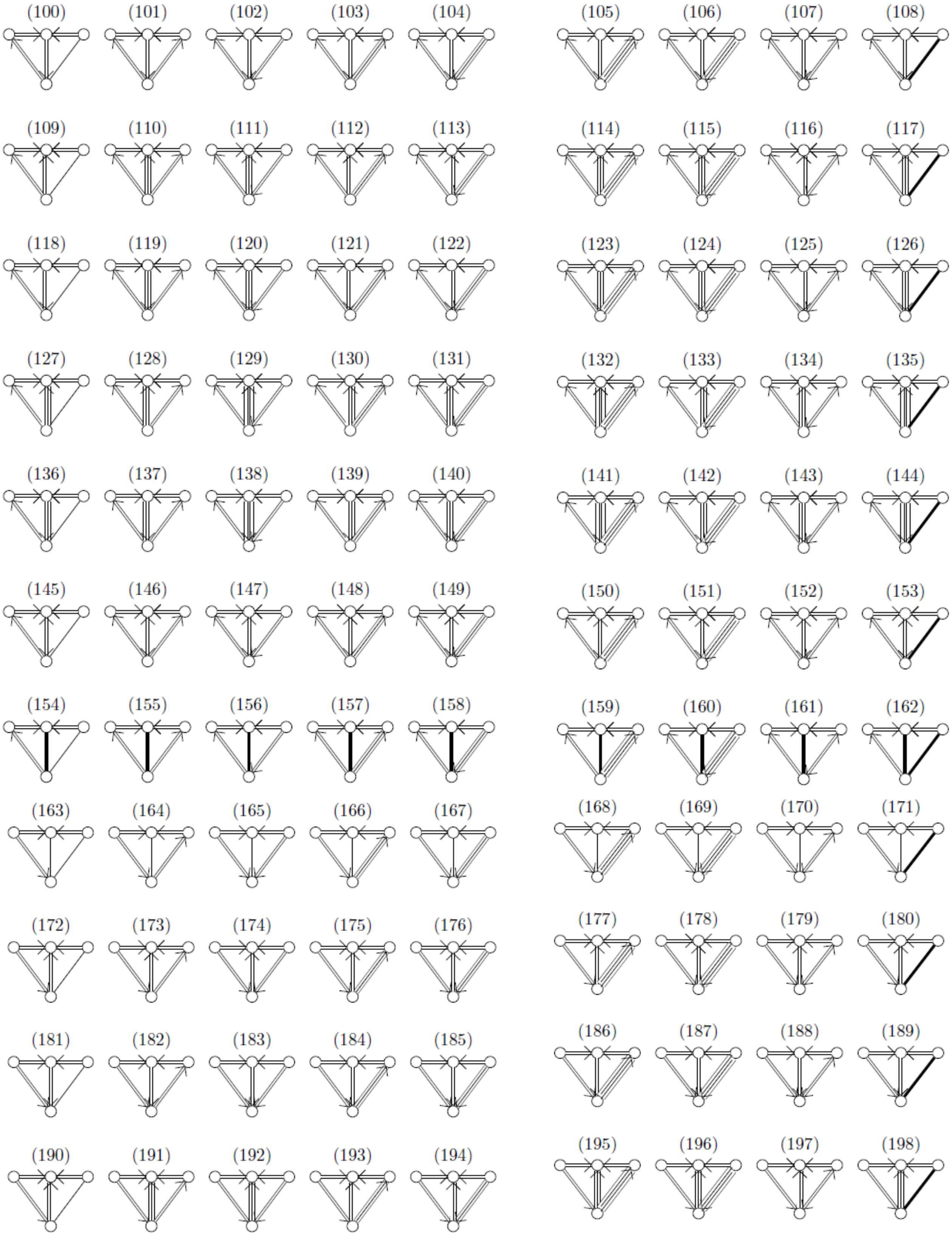
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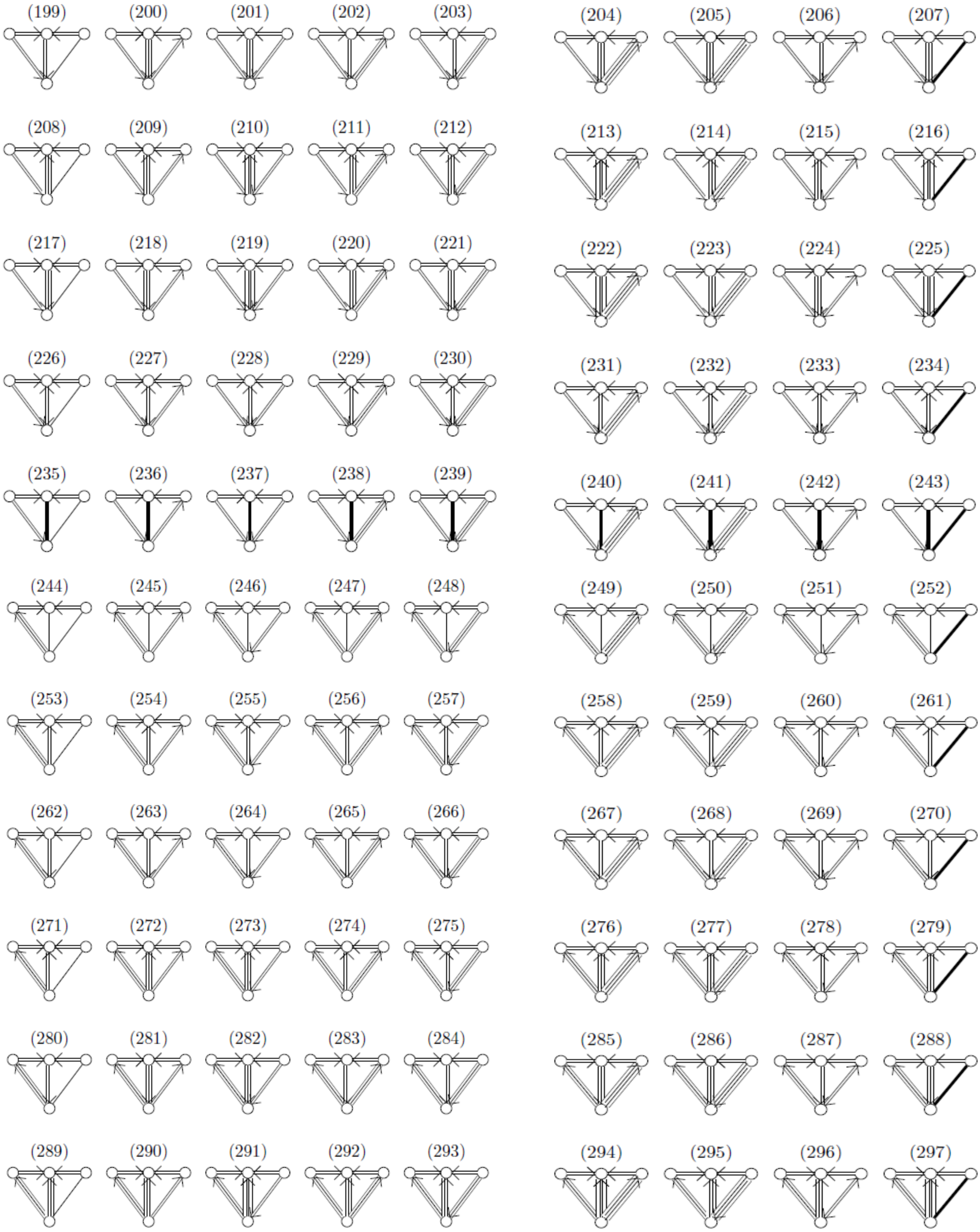
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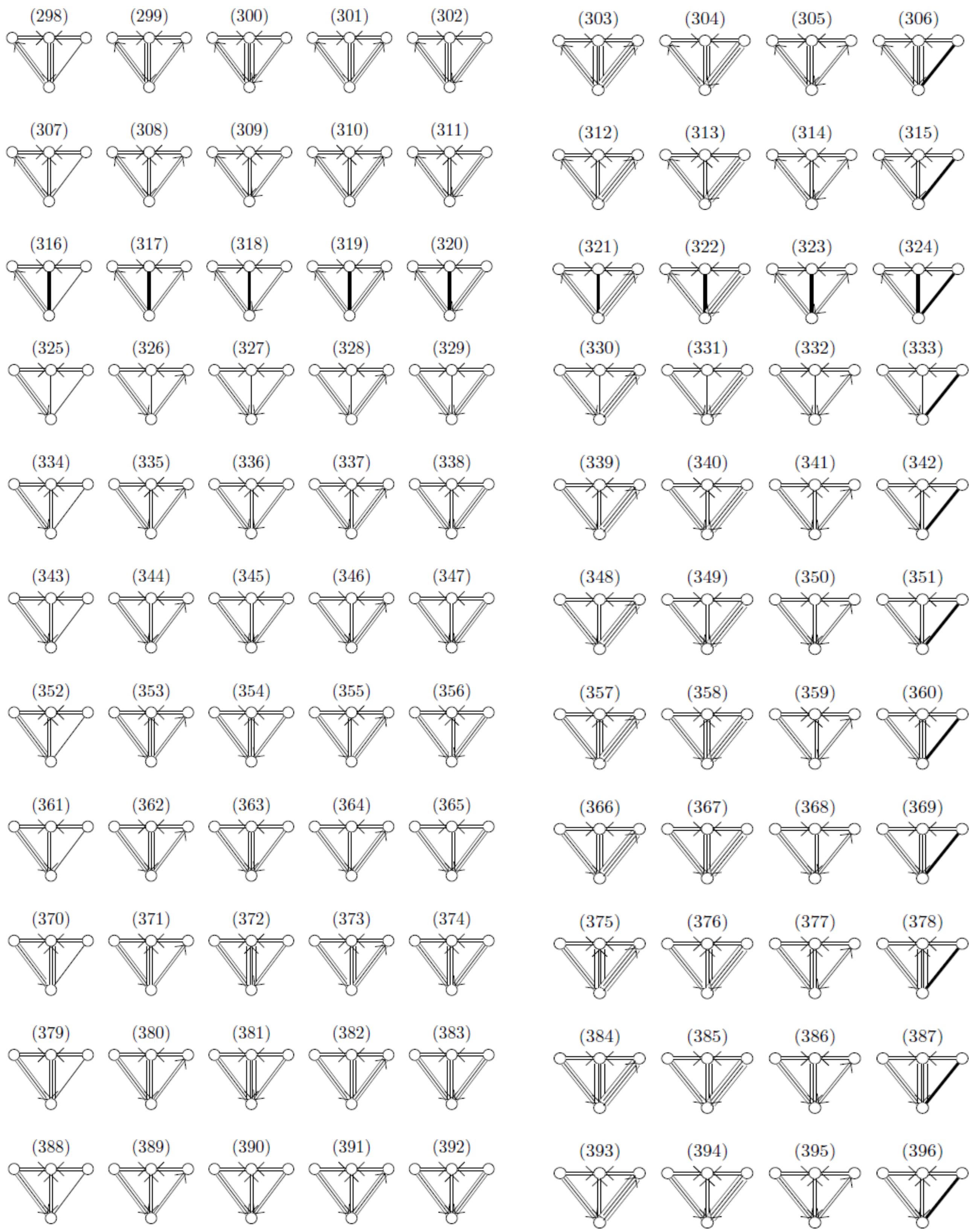
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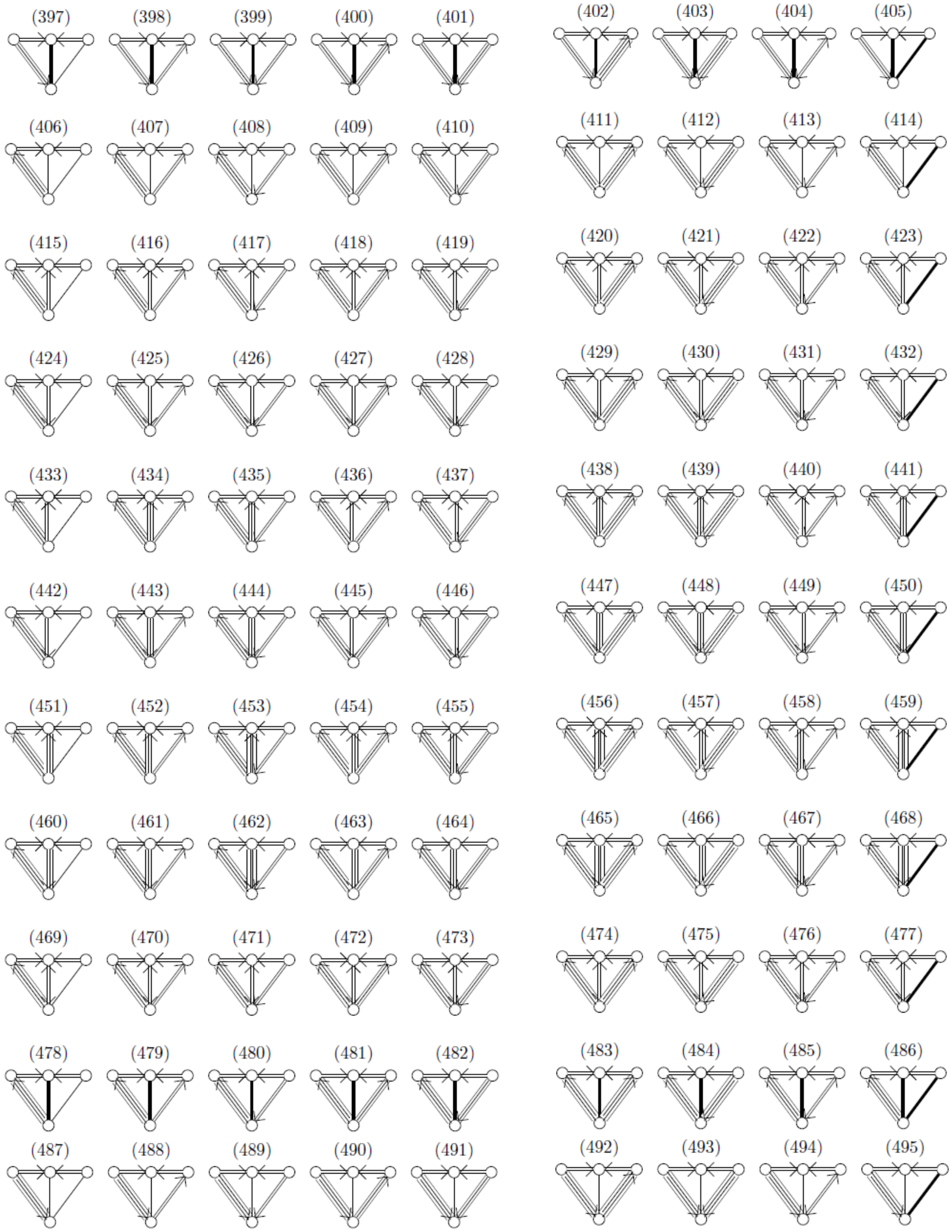
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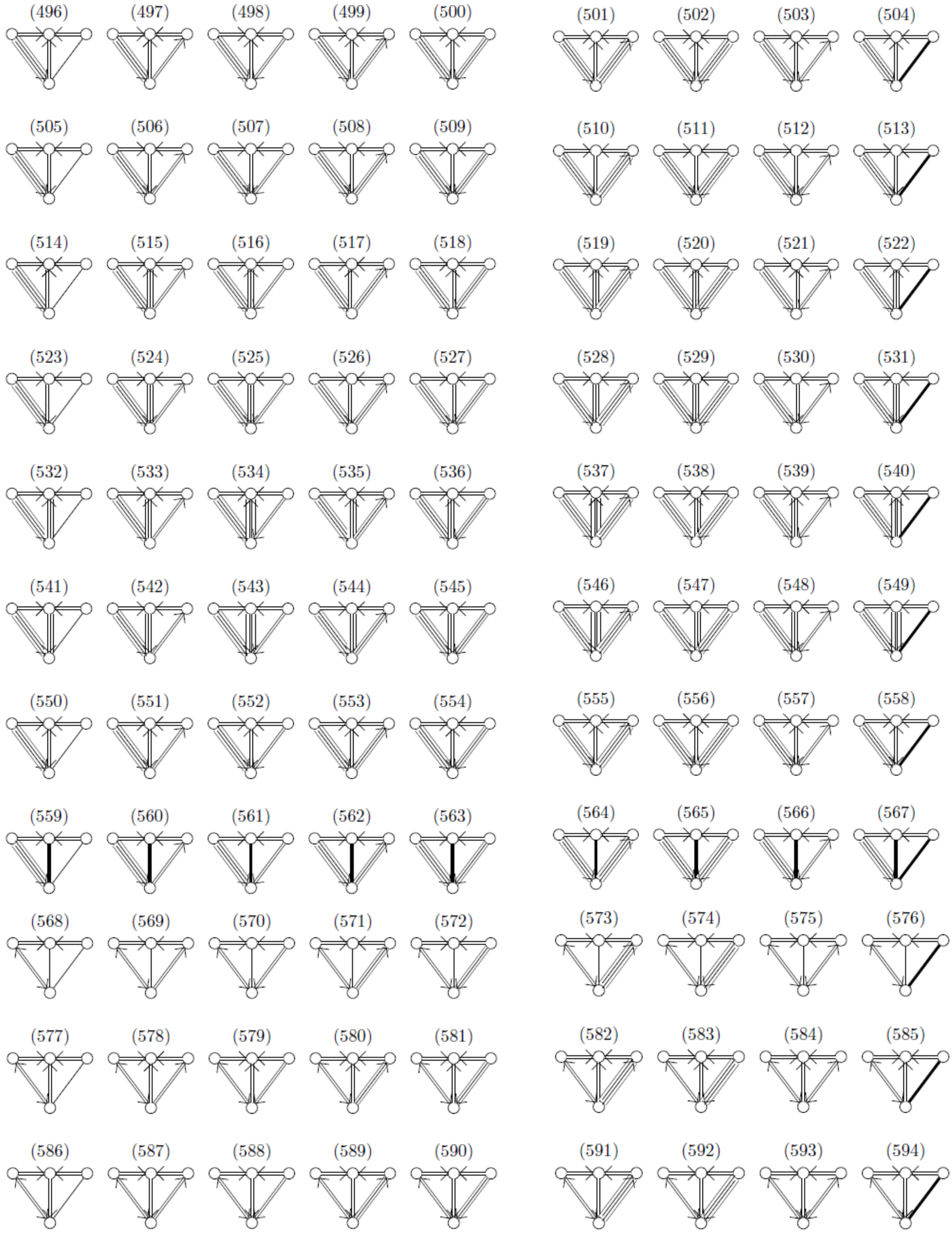
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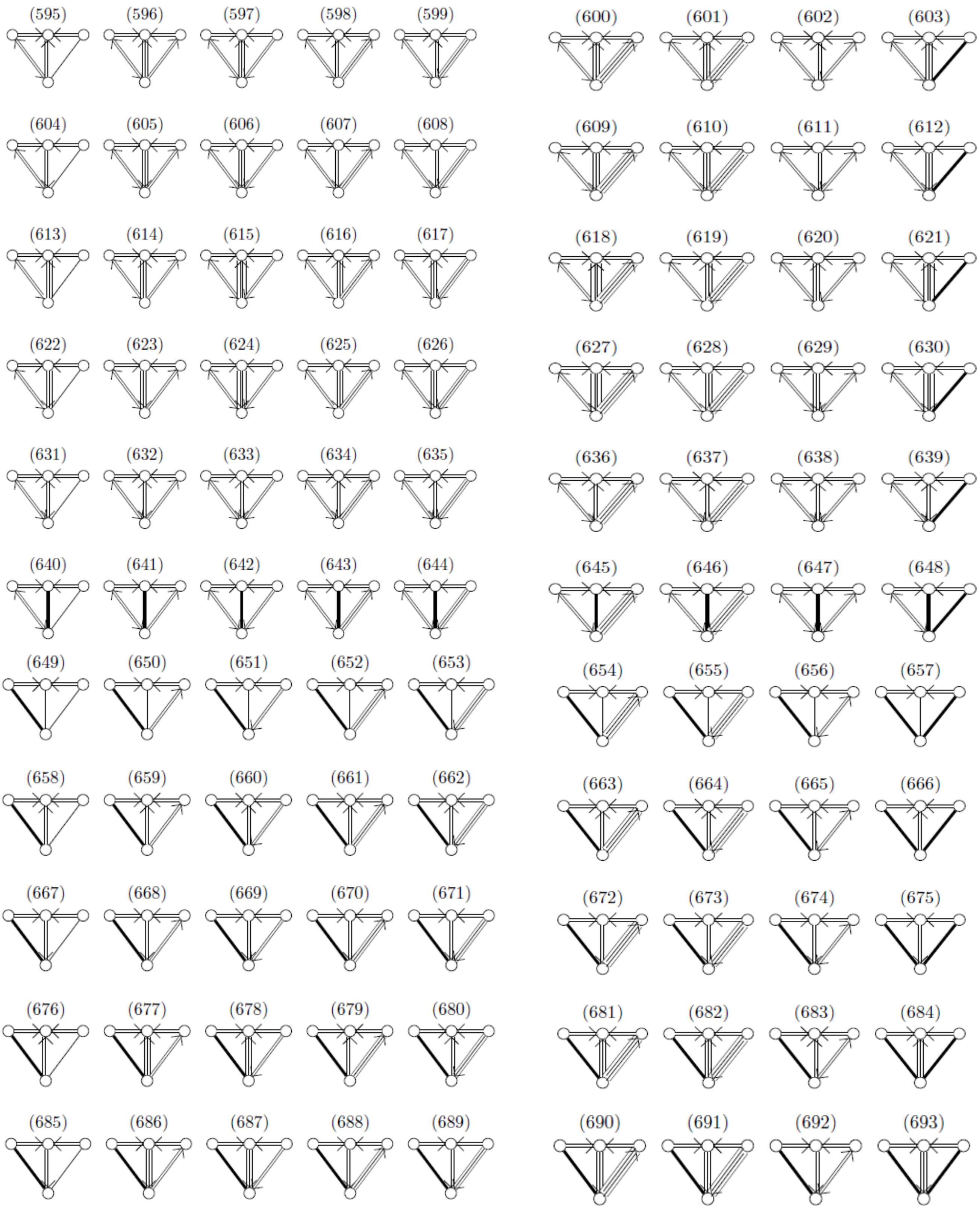
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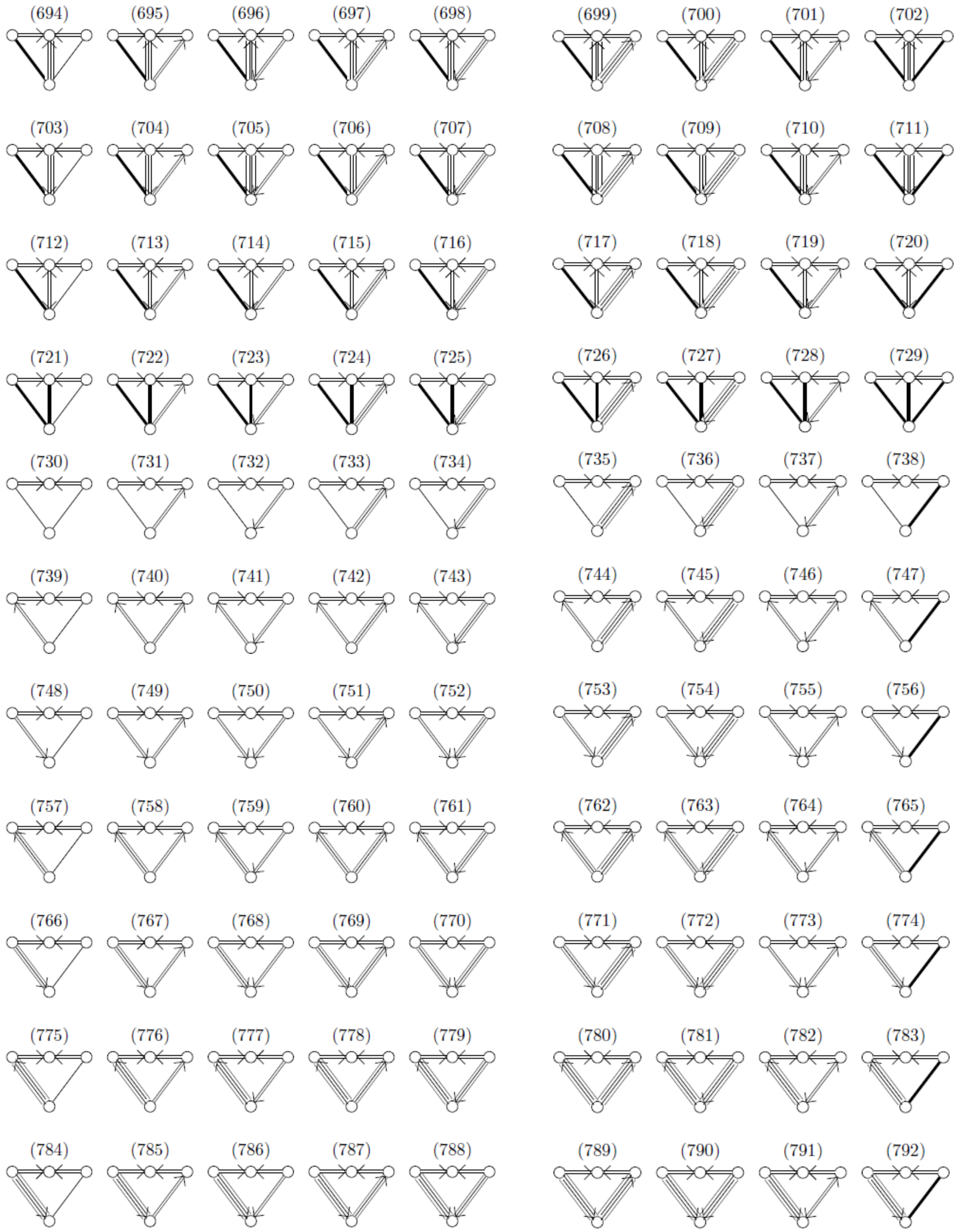
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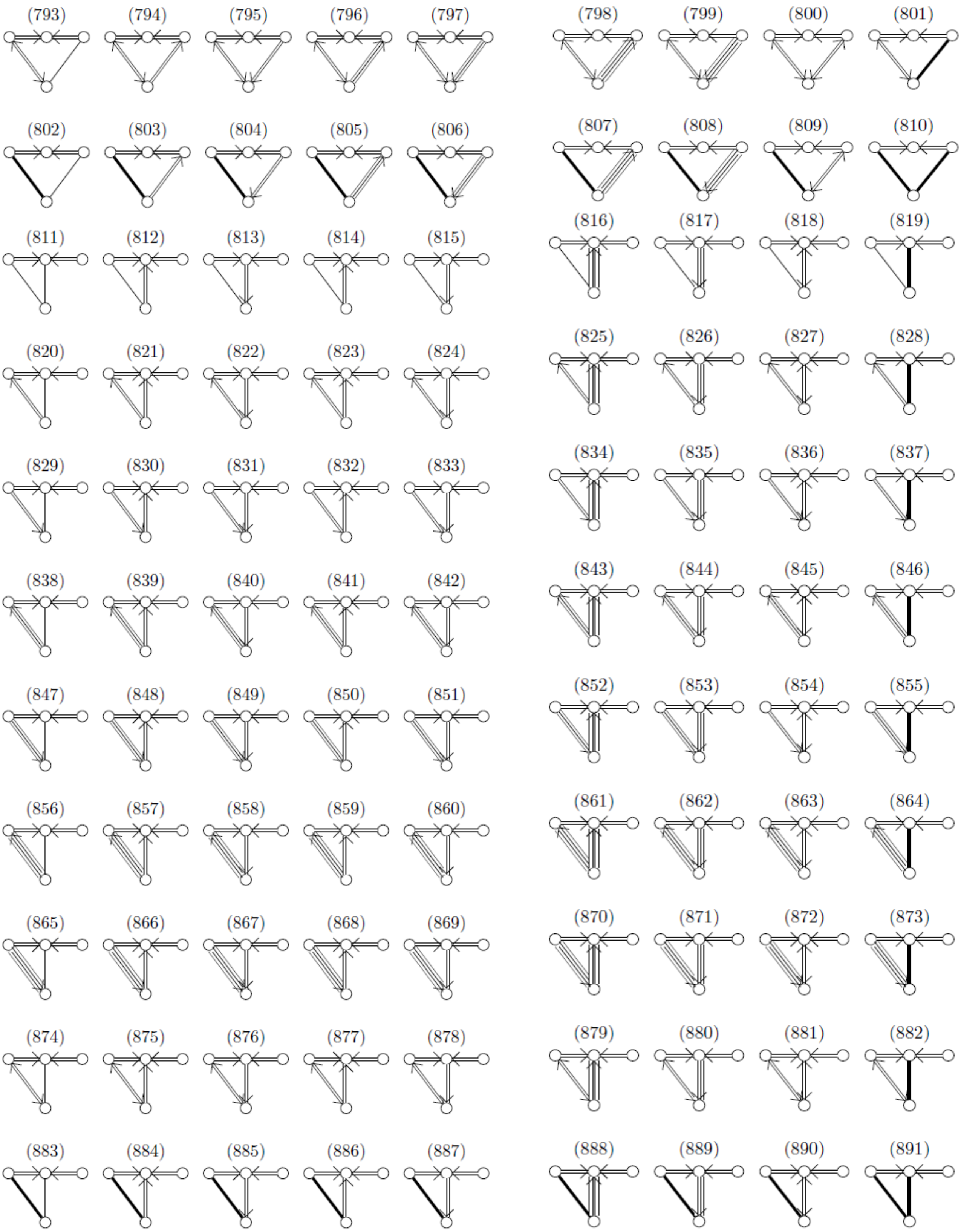
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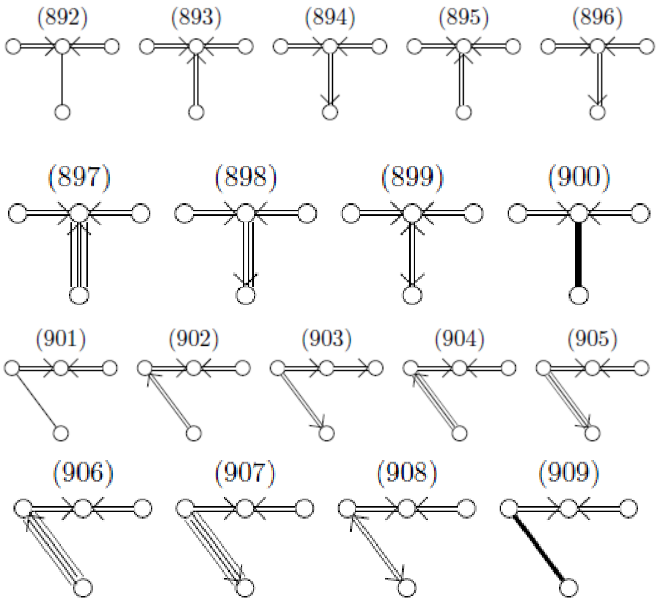


Proposition 1 : The quasi affine Kac Moody algebra $QAC_2^{(1)}$ satisfies the purely imaginary root.

Proof : From the characterization of Kac Moody algebras with the purely imaginary property (given in [10]), any indecomposable symmetrizable GCM up to order 4 satisfies the purely imaginary property. Hence all imaginary roots are purely imaginary in $QAC_2^{(1)}$.

Proposition 2 : Among the 909 Dynkin diagrams, 648 families are extended hyperbolic , 8 are hyperbolic type , 253 Dynkin diagrams in $QAC_2^{(1)}$ are indefinite, which are neither hyperbolic nor extended hyperbolic.

Proof : Any Dynkin diagram that contains a bold faced edge can not be of extended hyperbolic type; 253 diagrams contain bold faced edges and hence they are neither hyperbolic nor of extended hyperbolic type; Still they are indefinite type of Dynkin diagrams.

All other diagrams which do not contain a bold faced edge are either of extended hyperbolic or hyperbolic type, From the table it is clear that there are 8 diagrams of hyperbolic type, namely diagrams (892) – (896), (901)-(903), since each proper connected sub diagram is of finite or affine type.

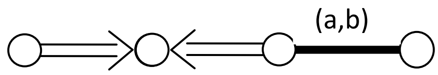
Leaving out these 8+253=261 diagrams, the remaining 648 Dynkin diagrams are all of extended hyperbolic type, since each proper connected sub diagrams in these cases are of finite, affine or hyperbolic types.

4 PROPERTIES OF ROOTS OF SPECIFIC CLASSES OF $QAC_2^{(1)}$

In this section, we consider three particular families of $QAC_2^{(1)}$ and identify the short and long real roots; Among the imaginary roots, we compute minimal imaginary and isotropic roots up to height 3 for these families;

Example 1 : Consider the GCM $A=\begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 2 & -2 & 0 \\ 0 & -1 & 2 & -a \\ 0 & 0 & -b & 2 \end{pmatrix}$,

whose associated Dynkin diagram is given by



where ab>5; Note that the GCM is symmetrizable, A=DB

$$=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & b/a \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & -1 & 0 \\ 0 & -1 & 2 & -a \\ 0 & 0 & -a & 2a/b \end{pmatrix};$$

The non generate symmetric bilinear form (,) defined on this algebra is given as follows:
$(\alpha_1,\alpha_1)=2$, $(\alpha_1,\alpha_2)=-1$, $(\alpha_2,\alpha_2)=1$, $(\alpha_2,\alpha_3)=-1$,
$(\alpha_3,\alpha_3)=2$, $(\alpha_3,\alpha_4)=-a$, $(\alpha_4,\alpha_4)=2a/b$
All fundamental roots are real .

Roots of height 2:
$(\alpha_1+\alpha_2, \alpha_1+\alpha_2)$ is a real root.
$(\alpha_2+\alpha_3, \alpha_2+\alpha_3)$ is a real root.
$(\alpha_3+\alpha_4, \alpha_3+\alpha_4) = 2+2a/b+2(-a)$
$\alpha_3+\alpha_4$ is imaginary if $1+a/b-a \le 0$
$\alpha_3+\alpha_4$ is isotrophic if $1+a/b-a = 0$
i.e. if $a+b-ab = 0$
$a+b = ab$ if and only if $a=b=2$
But the GCM is not quasi affine indefinite type if a=b=2
Therefore, this possibility is excluded;

To compute short and long real roots:

If a=b, then α_2 is the short root, and α_1, α_3, α_4 are long .
If a>b, then α_2 is the short root and α_4 is the long root.
If 2a=b, then α_2, α_4 are the short roots.
If 2a/b>2, then α_1, α_3 are the long roots.

Roots of height 3:
$(\alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3)= >0$; $\alpha_1+\alpha_2+\alpha_3$ can not be an imaginary root;
$(\alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4)= 1+2a/b-2a$
$\alpha_2+\alpha_3+\alpha_4$ is imaginary if $1+2a/b-2a \le 0$
$\alpha_2+\alpha_3+\alpha_4$ is not isotropic (For, If b=2, 2a+2=4a, a=1
a=1, b=2, the GCM is not quasi affine indefinite type).
Now,$(2\alpha_3+\alpha_4, 2\alpha_3+\alpha_4) = 4+2a/b-4a$
$2\alpha_3+\alpha_4$ is Imaginary if $4+2a/b-4a < 0$.

Example 2: Consider another family in $QAC_2^{(1)}$ whose

GCM is $\begin{pmatrix} 2 & -1 & 0 & 0 \\ -2 & 2 & -2 & -c \\ 0 & -1 & 2 & -a \\ 0 & -d & -b & 2 \end{pmatrix}$; The symmertizable decomposition is taken as

$$A=\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2d/c \end{pmatrix}\begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & 1 & -1 & -c/2 \\ 0 & -1 & 2 & -a \\ 0 & -c/2 & -a & c/d \end{pmatrix}$$

with cd,ab≠0 and the condition for symmetrizability is 2ad=bc ; The non generate symmetric bilinear form values are: $(\alpha_1,\alpha_1)=2$, $(\alpha_1,\alpha_2)=-1$, $(\alpha_2,\alpha_2)=1$, $(\alpha_2,\alpha_3)=-1$, $(\alpha_3,\alpha_3)=2$, $(\alpha_3,\alpha_4)=-a$, $(\alpha_4,\alpha_4)=c/d$, $(\alpha_2,\alpha_4)=-c/2$

Roots of height 2:

$(\alpha_1+\alpha_2, \alpha_1+\alpha_2)=1>0$; $\alpha_1+\alpha_2$ is a real root.
$(\alpha_2+\alpha_3, \alpha_2+\alpha_3)=1>0$ $\alpha_2+\alpha_3$is a real root.
$(\alpha_3+\alpha_4, \alpha_3+\alpha_4) = 2+(c/d)-2a$
$(\alpha_3+\alpha_4, \alpha_3+\alpha_4)$ is imaginary iff d+c-2a ≤ 0, i.e. iff c + d ≤ 2a,
$\alpha_3+\alpha_4$ is isotrophic if d+c-2a ≤ 0, this implies c + d ≤ 2a
$(\alpha_2+\alpha_4, \alpha_2+\alpha_4)=1+c/d-c$; $\alpha_2+\alpha_4$ is isotropic iff d+c-cd=0 ;
Thus $(\alpha_2+\alpha_4, \alpha_2+\alpha_4)$ is isotropic iff d+c-cd=0 iff c=d=2
Now $(\alpha_2+\alpha_4, \alpha_2+\alpha_4)$ is imaginary if d+c-cd ≤ 0;

Roots of height 3:

$(\alpha_1+\alpha_2+\alpha_3, \alpha_1+\alpha_2+\alpha_3)= 1>0$
$(\alpha_2+\alpha_3+\alpha_4, \alpha_2+\alpha_3+\alpha_4)= 1+c/d+-2a-c$
The root $\alpha_2+\alpha_3+\alpha_4$ is isotropic iff 1+c((1/d)-1)-2a=0
and this root is imaginary iff 1+c/d-2a-c≤0

Consider $(2\alpha_2+\alpha_4, 2\alpha_2+\alpha_4) = 4+c/d-c$
$2\alpha_2+\alpha_4$ is imaginary iff 4 ≤ c-(c/d)
$2\alpha_2+\alpha_4$ is isotropic iff 4 = c-(c/d);
Consider $(\alpha_2+2\alpha_4, \alpha_2+2\alpha_4) =1+(2c/d)-c$
$\alpha_2+2\alpha_4$, $\alpha_2+2\alpha_4$ is isotropic iff 1=c-2c/d
$\alpha_2+2\alpha_4$, $\alpha_2+2\alpha_4$ is imaginary iff 1 ≤ c-(2c/d)
Next, consider $(2\alpha_3+\alpha_4, 2\alpha_3+\alpha_4) = 4+c/d-2a$
$2\alpha_2+\alpha_4$ is imaginary iff 4 ≤ 2a-(c/d)
$2\alpha_2+\alpha_4$ is isotropic iff 4 = 2a-(c/d)
Now consider $(\alpha_3+2\alpha_4, \alpha_2+2\alpha_4) =2+(2c/d)-2a$;
$\alpha_2+2\alpha_4$ is isotropic iff 2=2a-2c/d
$\alpha_2+2\alpha_4$ is imaginary iff 2 < 2a-(2c/d)

Length of real roots : $|\alpha_i|^2=2, 1, 2, c/d$ for i=1,2,3,4 respectively;

If c=d, then α_2 and α_4 are the short root
If c/d< 2 then α_1, α_3 are the long root
If c/d=2 then α_1, α_3, α_4 are the long roots
If c/d >2 then α_4 is the long root.

Example 3: Consider the GCM

A=$\begin{pmatrix} 2 & -a & -b & -c \\ -a & 2 & -1 & 0 \\ -2b & -2 & 2 & -2 \\ -c & 0 & -1 & 2 \end{pmatrix}$,a,b,c are non negative integers not all 0; It is easy to check that A is a indecomposable symmetrizable GCM and the bilinear form is given by:

$(\alpha_1,\alpha_1)=2$, $(\alpha_1,\alpha_2)=-a$, $(\alpha_1,\alpha_3)=-b$ $(\alpha_2,\alpha_2)=2$, $(\alpha_2,\alpha_3)=-1$, $(\alpha_3,\alpha_3)=1$, $(\alpha_3,\alpha_4)=-1$, $(\alpha_1,\alpha_4)=-c$,$(\alpha_4,\alpha_4)=2$

Let $\alpha = k_1\alpha_1 + k_2\alpha_2 + k_3\alpha_3 + k_4\alpha_4$;

$$(\alpha,\alpha) = k_1\{2k_1 - ak_2 - bk_3 - ck_4\} + k_2\{-ak_1 + 2k_2 - k_3\} + k_3\{-bk_1 - k_2 + k_3 - k_4\} + k_4\{-ck_1 - k_3 + 2k_4\}.$$

Case 1: $k_3 = k_4 = 0$. Then $(\alpha,\alpha) = 2k_1^2 + 2k_2^2 - 2ak_1k_2$
If a > 2 & $k_1 = k_2$, $\alpha = k_1(\alpha_1+\alpha_2)$ is an imaginary root.

And,
$(\alpha_1+\alpha_2, \alpha_1+\alpha_2) = 2+2-2a = 4-2a < 0$, if a > 2
If a = 2, $(\alpha_1+\alpha_2)$ is imaginary and $(\alpha_1+\alpha_2, \alpha_1+\alpha_2)=0$ $\therefore (\alpha_1+\alpha_2)$ is an isotropic root.
Also, if $k_1^2 + k_2^2 - ak_1k_2=0$

$$\Rightarrow k_1 = \frac{ak_2 \pm \sqrt{a^2k_2^2 - 4k_2^2}}{2} = \frac{ak_2 \pm k_2\sqrt{a^2-4}}{2} = \frac{k_2}{2}\{a \pm \sqrt{a^2-4}\}$$

Any k_1 & k_2 satisfying the above condition will give an isotropic root $k_1\alpha_1 + k_2\alpha_2$.
Also, the minimal imaginary root is $\alpha_1+\alpha_2$ if a >1.

If $a > \frac{2k_1}{k_2}$, if $k_1 > k_2$, $\alpha = k_1\alpha_1 + k_2\alpha_2$ is imaginary.

Case 2: $k_2 = k_3 = 0$. Then $(\alpha,\alpha) = 2(k_1^2 + k_4^2) - 2ck_1k_4$
If a>1 & $k_1 = k_4$, $\alpha = k(\alpha_1+\alpha_4)$ is imaginary.

If $a > \frac{2k_4}{k_1}$, if $k_1 < k_4$, $\alpha = k_1(\alpha_1+\alpha_4)$ is imaginary.

For $a > \frac{2k_4}{k_1}$, if $k_1 < k_4$, α is imaginary.

$$(\alpha_1+\alpha_4,\alpha_1+\alpha_4)=2+2+2(-c)=4-2c<0, \text{if } c>1$$
$$\therefore \alpha_1+\alpha_4 \text{ is minimal imaginary root if } c>1.$$

$\alpha_1+\alpha_4$ is isotropic if c =2.
If $k_1^2+k_4^2-ak_1k_4=0$

$$\Rightarrow k_1=\frac{ck_4\pm\sqrt{c^2k_4^2-4k_4^2}}{2}=\frac{k_4}{2}\{c\pm\sqrt{c^2-4}\}$$

If k_1 & k_4 satisfy this relations then $k_1\alpha_1+k_4\alpha_4$ will be isotropic.

Case 3: $k_2 = k_4 = 0$; If b > 3/2 & $k_1 = k_3$, Then
$\alpha=k_1\alpha_1+k_3\alpha_3$ is an imaginary root.

If $\frac{3k_3}{2k_1}<b$ & $k_1<k_3$, α is imaginary.

If $\frac{3k_1}{2k_3}<b$ & $k_3<k_1$, α is imaginary.

$$(\alpha_1+\alpha_3,\alpha_1+\alpha_3)=3-2b;$$
If $b>3/2$, $\alpha_1+\alpha_3$ is minimal imaginary root.

5. CONCLUSION

In this work, classification of Dynkin diagrams and some properties of roots are obtained for the family $QAC_2^{(1)}$. Further , using the representation theory , the dimensions of the root spaces and weight spaces can be computed.

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