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A Concept of Notional Frequency of an Aperiodic Forcing Function to Facilitate Discernment of Resonance Caused by Modal Coincidence with the Driven System

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Abstract - *A concept of notional frequencies (or modes) of an aperiodic function of time is presented. These notional frequencies emerge as the roots of a 'Companion Polynomial', which latter is constructed such as to annihilate (and also be annihilated by) the given function of time, of which it is the companion. A method to obtain this polynomial is suggested, using which the companion polynomial of certain functions of the general exponential class and of their polynomial-modulated forms have been derived. The notional modes of the said functions of time are then extracted from the so derived polynomials. This concept and the attendant method has been put to use in demonstrating the resonant behaviour of first-order electrical circuits subjected to forcing functions with such notional modes as to match the natural modes of the circuits upon which they are impressed. The complete solution of the circuit under such resonant conditions has been presented. The generality of the concept of resonance has been thus emphasized and its applicability aptly extended to include aperiodic forcing functions as well. A facile means of recognizing circuit-resonance, in its most general form – by virtue of a simple examination of the all the participating modes - is thus placed in evidence.*

***Key Words*:** Resonance, notional frequency, modal coincidence, aperiodic forcing function, natural modes, electrical circuit theory.

1.INTRODUCTION

There exists a fairly widespread sense of agreement upon the general meaning of the term 'Resonance', with regard to system analysis spanning a large variety of engineering disciplines. It being: the particular phenomenon arising from a certain manner of concordance between the driving forces and the driven system. The etymology as well as the early technical use of this term indicate its origins in the science of acoustics, wherein it is understood to mean a 'sympathetic' reinforcing of the sound waves of matching frequencies (or modes). This meaning is generally held to be analogously true in other fields as well. Herefrom arises the cogent idea of a 'match' or a 'coincidence' between/among the 'modes' or 'frequencies' (in some specific sense of these words) as being the cause of the phenomenon of resonance. The terms 'Modal matching' or 'Modal coincidence' find currency on the above considerations, and are taken to connote what they denote, quite literally, as they should be.

However, in respect of electrical circuit theory, the term 'Resonance' has been used in a rather restrictive manner, and is sometimes used to denote phenomena that do not merit such a usage. A careful study of some of the classic and current works on circuit theory [1-30] attests to this observation. There is usually a tendency to fixate resonance with either one kind of signal or with some specific network configurations. Attempts to apply and appreciate the plenary sense of this term in circumstances specific to electrical circuit theory have been routinely frustrated by numerous obfuscations and (unfortunately) deeply-entrenched perceptions, some of which are listed below:

1) Depiction of resonance as being somehow specific and exclusive to the sinusoidal steady-state.
2) Treatment of the unity-power-factor (upf) conditions in certain specific network configurations (series/parallel RLC or combinations thereof) as instances of resonance and proceeding forth to define the 'resonant' frequency, in such a manner as to denote conditions of mere 'cresting' (or 'troughing') in the swept-radian-frequency domain.

3) Portrayal of certain pole-zero cancellations as manifestations of an 'all-frequency' resonance.
4) An erroneous consideration of the natural response of the second order lossless LC tank as an occurrence of resonance.
5) A baseless assertion of necessitating the conjoint presence of an inductor and a capacitor for the occurrence of resonance.

Whereas each of the above perceptions is misleading, they may be overcome with the power of reason. Nevertheless, one may still find oneself stranded for want of a tangible measure of the 'frequency' of a general function of time, be it periodic or not, for the purpose of comparison with the natural frequencies of the driven system.. This paper attempts to supply such a need by proposing a 'notion' of frequency or a 'Notional Frequency' of any mathematically specifiable function of time, and by suggesting a method wherewith to compute this metric for a given function of time. With this quantification, the process of identifying resonance conditions in a given system driven by a forcing function of a specific description is reduced to a comparison of the natural modes of the system with the 'notional' modes of the forcing function. Should even one match be found, the phenomenon of resonance occurs. Should there be more than one match, a multi-modal resonance instances itself. If there are no matches at all, a normal non-resonant response ensues. This concept has been illustrated in this paper through examples of resonant phenomena occurring in first-order electrical circuits driven by aperiodic functions bearing elementary mathematical descriptions. The simplicity of the systems employed herein for demonstration must not be construed to imply triviality of the propounded concept. The concept and method presented here stand applicable to systems of any order and complexity.

2. CHARACTERISTIC POLYNOMIAL AND NATURAL MODES OF A SYSTEM

The mathematical operation of differentiation with respect to time is denoted [31] by the operator p as in,

$$p \equiv \frac{d}{dt} \qquad (1)$$

For a first-order system described by

$$px = ax + b\,;\ x|_{t=0} = x(0) \qquad (2)$$

D(p) = (p-a) represents the characteristic polynomial and its only root $p_1 = a$ is its natural mode (or frequency). p_1 is a real number and is specified in units of nepers per second (nep/s or Np/s)or just s^{-1}.

b represents the forcing function. The natural response of this system may be obtained by setting $b = 0$. This condition is referred to as 'passivation' (or 'passivization'), under which conditions, the response (natural) of the system described in (2) is given by

$$x = e^{at}\,x(0) \qquad (3)$$

2.1 Test System 1: First-Order Lossless System

The test system along with its passivated form is as shown in Fig-1. It has one degree of freedom [4], expressible in the energy-indicating variable (or state variable) i_L.

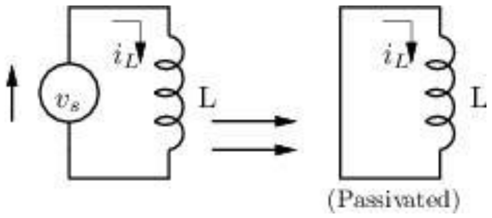

Fig-1: Test System 1: A first-order lossless system and its passivated form

The governing differential equation of this system is

$$pi_L = (0)i_L + \frac{v_s}{L} \qquad (4)$$

It is evident that the characteristic polynomial is $D(p) = p$ and that the natural mode of the system is $p_1 = 0$. (This is true of any first-order lossless system.)

The natural response of this system is given by

$$i_L = i_L(0)\,;\, t \geq 0 \qquad (5)$$

2.2 Test System 2: First-Order System with Dissipation

The test system along with its passivated form is as shown in Fig-2. It has one degree of freedom, expressible in the energy-indicating variable (or state variable) i_L.

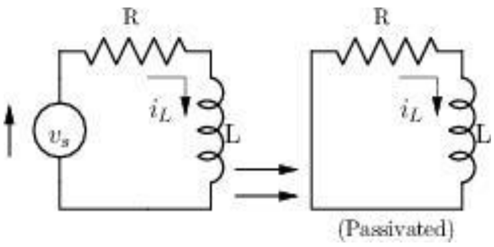

Fig-2: Test System 2: A first-order system with dissipation and its passivated form

The governing differential equation is

$$pi_L = -\frac{R}{L}i_L + \frac{v_s}{L} \qquad (6)$$

It is evident that the characteristic polynomial is $D(p) = \left(p + \frac{R}{L}\right)$ and that the natural mode of the system is

$p_1 = -\left(\frac{R}{L}\right)$ nep/s. The natural response of the above system is given by,

$$i_L = e^{-\left(\frac{R}{L}\right)t} i_L(0);\ t \geq 0 \qquad (7)$$

3. NOTIONAL MODES OF A FUNCTION $\Psi(t)$

3.1 Definition

Let $\Psi(t)$ be a general 'singly-describable' function of time; that is, let $\Psi(t)$ have a single mathematical description for all t. No periodicity is necessarily implied in this description, while no attempt is made to rule it out either.

Let there exist a polynomial $C(p)$ with real coefficients, composed with the operator p as follows:

$$C(p) = c_0 + c_1 p + c_2 p^2 + \cdots + c_m p^m \qquad (8)$$

This polynomial $C(p)$ is an 'operational polynomial' or a 'polynomial operator', which when operating on any function $f(t)$, produces another function which is the weighted summation of $(m+1)$ derivatives (the zero-order derivative included).

$$C(p)[f(t)] = (c_0 + c_1 p + c_2 p^2 + \cdots + c_m p^m)\ [f(t)]$$

$$C(p)[f(t)] = \sum_{k=0}^{k=m} c_k \frac{d^k}{dt^k}[f(t)]$$

Let $C(p)$ be so conceived that

$$C(p)[\Psi(t)] = 0 \qquad (9)$$

Such a polynomial operator $C(p)$ annihilates the function $\Psi(t)$. Equivalently, the function $\Psi(t)$ satisfies the operational polynomial $C(p)$. This companionship or concomitance is mutual, and the polynomial $C(p)$ may therefore be termed as a 'Companion' to the function $\Psi(t)$ (and vice versa).

Every value of p for which $C(p) = 0$, or every root of the polynomial $C(p)$ is a 'notional frequency' or a 'notional mode' of the companion function $\Psi(t)$. This term is employed to suggest that the frequency (or equivalently the period), is not necessarily actual. These notional modes are quantifiable in a manner identical with that of the natural modes. The number of the notional modes for a given $\Psi(t)$ will be the same as the degree of $C(p)$. As with the roots of any polynomial (with real coefficients), these notional frequencies could be real, imaginary, or complex.

3.2 Examples of Obtaining the Companion Polynomial

Given a function $\Psi(t)$, its companion $C(p)$ is to be sought as the barest minimal polynomial form which causes the annihilation of $\Psi(t)$. Having obtained this form of $C(p)$, its roots could then be extracted to yield the notional frequencies of $\Psi(t)$.

In what follows, a companion $C(p)$ has been obtained for four choices of $\Psi(t)$. These forms of the function $\Psi(t)$ have been so chosen as to be the forms of forcing functions capable of causing resonance in the two test systems depicted; that is, so chosen as to have their notional frequencies coinciding with the natural modes of the test systems. (The test systems 1 and 2 have natural modes of 0 and $-\left(\frac{R}{L}\right)$ respectively.)

In each of the following cases, the companion polynomial $C(p)$ is found by one or two successive differentiations of the form of $\Psi(t)$ and suitable algebraic manipulations thereafter.

1) *Forms of $\Psi(t)$ Capable of Causing Resonance in Test System 1:* The test system 1 has a natural mode of 0 nep/s. To be able to cause resonance in this system, the forcing function must also have a notional frequency of 0 nep/s. The following forms have been considered (m_0, m_1 are real constants):

1] $\Psi(t) = m_0$, a non-zero constant.

2] $\Psi(t) = m_1 t$, a linear-modulated constant (linear ramp).

Table 1 shows the companion pairs $\Psi(t)$ and $C(p)$ for these two cases.

Table-1: Companion Pairs $\Psi(t)$ and $C(p)$: The Constant (Zero-Exponential) Function and Linear Modulation thereof

Case	Forcing Function $\Psi(t)$	Companion Polynomial $C(p)$	Notional Frequencies of $\Psi(t)$ from $C(p) = 0$ (nep/s)
1. a)	$\Psi(t) = m_0$	$C(p) = p$	$p_1 = 0$
b)	$\Psi(t) = m_1 t$	$C(p) = p^2$	$p_i = 0$, $i = 1,2$

2) *Forms of $\Psi(t)$ Capable of Causing Resonance in Test System 2:* The test system 2 has a natural mode of $-\left(\frac{R}{L}\right)$ nep/s. To be able to cause resonance in this system, the forcing function must have a notional frequency of the same value. The following forms have been considered (m_0, m_1 are real constants):

1] $\Psi(t) = m_0 e^{\sigma t}$, a real exponential, with $\sigma = -\left(\frac{R}{L}\right)$

2] $\Psi(t) = m_1 t e^{\sigma t}$, a linear-modulated real exponential, with $\sigma = -\left(\frac{R}{L}\right)$.

(If $\sigma \neq -\left(\frac{R}{L}\right)$, no modal coincidence would take place and a normal non-resonant response results.) Table 2 shows the companion pairs $\Psi(t)$ and $C(p)$ for these two cases.

Table-2: Companion Pairs $\Psi(t)$ and $C(p)$: The Real-Exponential Function and Linear Modulation thereof

Case	Forcing Function $\Psi(t)$	Companion Polynomial $C(p)$	Notional Frequencies of $\Psi(t)$ from $C(p) = 0$ (nep/s)
2. a)	$\Psi(t) = m_0 e^{\sigma t}$	$C(p) = p - \sigma$	$p_1 = \sigma$
b)	$\Psi(t) = m_1 t e^{\sigma t}$	$C(p) = (p - \sigma)^2$	$p_i = \sigma$, $i = 1,2$

An illustration of creation of an entry in the Table 2, corresponding to the case (2.b) is shown below.

With $\Psi(t) = m_1 t e^{\sigma t}$,

$$p\Psi = m_1 e^{\sigma t} + \sigma[m_1 t e^{\sigma t}]$$

i.e. $p\Psi = m_1 e^{\sigma t} + \sigma\Psi$

$$(p - \sigma)\Psi = m_1 e^{\sigma t}$$

Whence,

$$(p - \sigma)^2\Psi = (p - \sigma)(m_1 e^{\sigma t}) = 0$$

Thus $C(p) = (p - \sigma)^2$ and $p_{1,2} = \sigma$ nep/s. (Two real notional modes at σ.)

4. RESONANT SITUATIONS IN FIRST-ORDER SYSTEMS

Two first-order electrical circuits are chosen for exemplification of the phenomenon of resonance. Resonant situations are created by impressing a source (forcing function) of such a description as to have its notional mode(s) coinciding with that of the circuit. Arbitrary initial conditions could be assumed. The governing differential equation is formulated and the unique solution of the resulting initial-value problem is presented.

4.1. First-Order Lossless System

The test system 1 (Fig-1) is placed under consideration. The only natural mode of this system is 0 nep/s. Two cases of resonant conditions are caused in this system by impressing forcing functions possessing such notional frequencies as to match the natural mode of the system.

1) *Constant excitation:* Let the description of the voltage source be $v_s = m_o$, a constant. The only notional mode associated with ($v_s = m_o$) is zero (Table 1). There is thus a resonant condition due to modal coincidence. The unique solution for i_L in this case is given by

$$i_L = i_L(0) + \left[\frac{m_o}{L}\right] t \; ; t \geq 0 \qquad (10)$$

The waveforms of the source voltage v_s and of the inductor current i_L pertaining to this case have been shown in Fig-3.

2) *Linear ramp excitation:* Let the description of the voltage source be $v_s = m_1 t$. The two notional modes associated with ($v_s = m_1 t$) are zero (twice) (Table 1). There is thus a resonant condition due to modal coincidence. The unique solution for i_L in this case is given by

$$i_L = i_L(0) + \left[\frac{m_1}{2L}\right] t^2 \; ; t \geq 0 \qquad (11)$$

The waveforms of the source voltage v_s and of the inductor current i_L pertaining to this case have been shown in Figure 4.

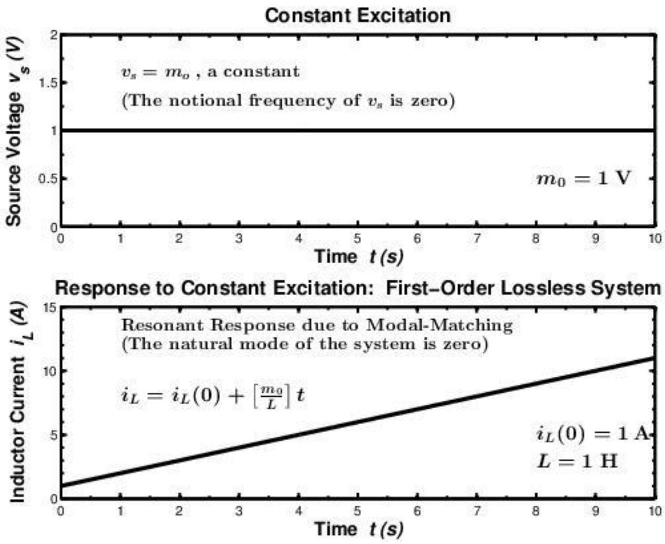

Fig-3: Resonant response of a first-order lossless system subjected to a constant excitation

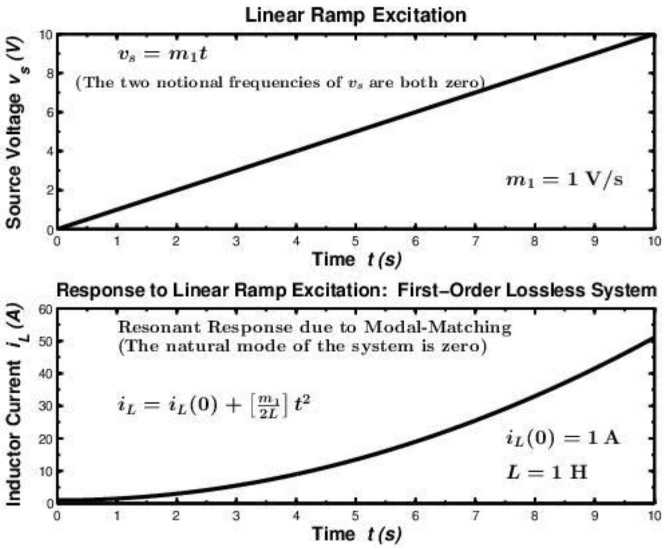

Fig-4: Resonant response of a first-order lossless system subjected to a linear ramp excitation

4.2. First-order system with dissipation

The test system 2 (Fig-2) is placed under consideration. The natural mode of this system is $-\left(\frac{R}{L}\right)$ nep/s. Two cases of resonant conditions are caused in this system by impressing forcing functions possessing such notional frequencies as to match the natural mode of the system.

1) *Exponential excitation:* Let the description of the voltage source be $v_s = m_0 e^{-\frac{R}{L}t}$. The only notional mode associated with this is v_s *is* $-\left(\frac{R}{L}\right)$ nep/s (Table 2). There is thus a resonant condition due to modal coincidence. The unique solution for i_L in this case is given by

$$i_L = e^{-\frac{R}{L}t} i_L(0) + \left[\frac{m_0}{L}\right] t e^{-\frac{Rt}{L}}\,;\ t \geq 0 \qquad (12)$$

The waveforms of the source voltage v_s and of the inductor current i_L pertaining to this case have been shown in Fig-5.

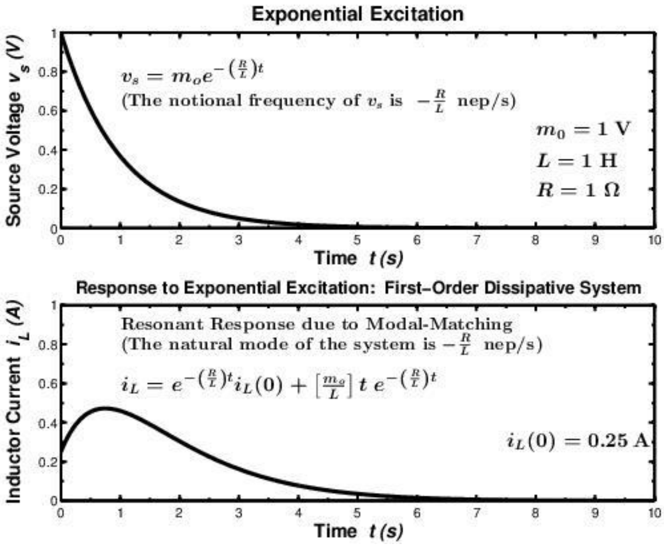

Fig-5: Resonant response of a first-order dissipative system subjected to an exponential excitation

2) *Linear-modulated exponential excitation:* Let the description of the voltage source be $v_s = m_1 t\, e^{-\frac{Rt}{L}}$. The two notional modes associated with this v_s are $-\left(\frac{R}{L}\right)$ nep/s (twice) (Table 2). There is thus a resonant condition due to modal coincidence. The unique solution for i_L in this case is given by

$$i_L = e^{-\frac{R}{L}t} i_L(0) + \left[\frac{m_1}{2L}\right] t^2 e^{-\frac{Rt}{L}}\,;\ t \geq 0 \qquad (13)$$

The waveforms of the source voltage v_s and of the inductor current i_L pertaining to this case have been shown in the Fig-6.

4.3. Summary of the Four Resonant Cases

The expressions for i_L which have emerged as the solutions of the four analyzed cases of first-order resonance are summarized in Table 3.

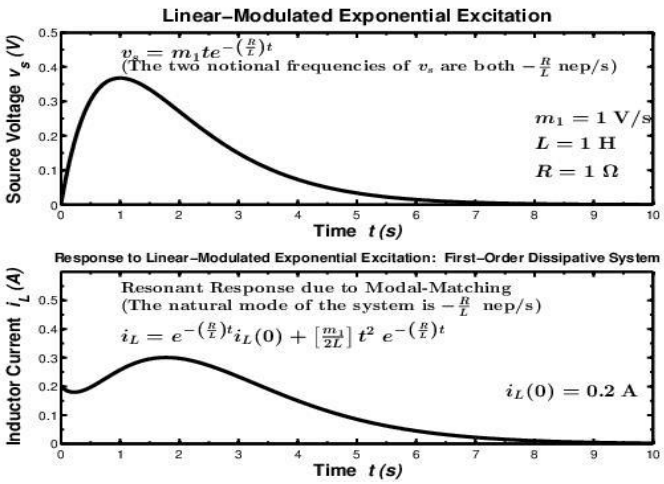

Fig-6: Resonant response of a first-order dissipative system subjected to a linear-modulated exponential excitation

Table-3: Results Depicting the Excitation v_s and the Response i_L under the Resonant Conditions Analyzed

Case	Excitation $v_s(t)$	Resonant Response $i_L(t)$
1. a)	$v_s = m_0$	$i_L = i_L(0) + \left[\frac{m_0}{L}\right] t$
1. b)	$v_s = m_1 t$	$i_L = i_L(0) + \left[\frac{m_1}{2L}\right] t^2$
2. a)	$v_s = m_0 e^{-\frac{R}{L}t}$	$i_L = e^{-\frac{R}{L}t} i_L(0) + \left[\frac{m_0}{L}\right] t e^{-\frac{Rt}{L}}$
2. b)	$v_s = m_1 t e^{-\frac{R}{L}t}$	$i_L = e^{-\frac{R}{L}t} i_L(0) + \left[\frac{m_1}{2L}\right] t^2 e^{-\frac{Rt}{L}}$

5. SUMMARY AND CONCLUSION

Although a first order system was treated in the foregoing discussion, the general essence of the exposition remains unchanged for LTI lumped-parameter systems of any order. The first-order system was specifically chosen for its simplicity and the resulting ease of illustration. This choice was also a deliberate one – to secure a release from the customary fixation of the phenomenon of resonance with a second-order system. Resonance could occur whenever there happens to be a coincidence of the participating modes – natural modes of the driven system, and the notional frequencies of the forcing function – regardless of the order of the system. The concept of notional frequency aids in the recognition of such a state of affairs.

The following remarks may be made from the case-studies presented:

1) A resonant condition does not have to necessarily cause unbounded growth of the state variable(s). There is indeed an unbounded growth in the case of lossless systems. Boundless increase can also result if the nature of the forcing function (to cause resonance) is in itself such as to increase in an unbounded manner with time. (Such as in the case of a modal match on the right-half of the number plane.)
2) A Linear-modulation does not bring in newer notional frequencies, but causes the existing notional modes to be replicated.
3) A system would resonate if a forcing function has the form of the system's own natural response or a component thereof. This is how the 'sympathetic' driving causes reinforcement of the natural response.
4) The process of obtaining the companion polynomial $C(p)$ for a given function $\Psi(t)$ is analogous to the process of synthesizing a network whose natural response is $\Psi(t)$, or whose characteristic polynomial is $C(p)$.

As an end-note, it may be mentioned that a sinusoid $\sin(\omega t+\lambda)$ could cause resonance in a second-order system with natural frequencies of $\pm j\omega$ rad/s; that is, only in a lossless LC combination of element values such that $\frac{1}{\sqrt{LC}}=\omega$ rad/s. RLC circuits can never resonate with a sinusoid. They would require a forcing function of the $e^{\sigma t}\sin(\omega t)$ description to do so.

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