Existence result of solutions for a class of generalized p-laplacian systems

R.Sahandi Torogh

Department of Mathematics, Varamin-Pishva Branch, Islamic Azad University, Varamin, Iran

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Abstract: In this paper, I study the existence of solutions to a class of nonlinear problem. I generalize the existence results of a class of p-Laplacian equation and extend it to a (p,q)-Laplacian system. Using some theorems, I establish sufficient conditions under which, this problem can be solved.

Keywords: solution, non-linear equation, (p,q)-Laplacian system

1. INTRODUCTION

In recent years, BVP has received a lot of attention. In [1] the authors have studied the existence of solution to the nonlinear p-Laplacian equation

$$-\left(r^{n-1}|u'|^{p-2}u'\right)' = r^{n-1}\psi(r)f(u), on\ [0,1] \quad (1.1)$$

$$u'(0)=0, u(1)=0 \quad (1.2)$$

when $r=|x|, x\in\Omega\subset\mathbb{R}^n$, $\psi\in C^1(\mathbb{R}^+)$.

We introduce the following eigenvalue problem

$$-\left(r^{n-1}|u'|^{p-2}u'\right)' = \lambda r^{n-1}\psi(r)|u|^{p-2}u\ , on\ [0,1] \quad (1.3)$$

$$u'(0)=0, u(1)=0,$$

In [3] the authors proved that (1.3) has a countable number of eigenvalue $\{\lambda_i\}_{i\in\mathbb{N}}$ satisfying $\lambda_i<\lambda_j$, when $i<j$, $\lim_{i\to+\infty}\lambda_i=\infty$ and the corresponding eigenfunction $u_k(r)$ has exactly $k-1$ zero in (0,1).

2. Preliminaries and Lemmas

In this section, we state some theorem according to the references.

Consider $u(0)=\alpha, u'(0)=0, (*)$

In this paper, I study the following system:

$$\begin{cases} -\left(r^{n-1}|u'|^{p-2}u'\right)' = r^{n-1}A(r)f(u,v) \\ -\left(r^{n-1}|v'|^{p-2}v'\right)' = r^{n-1}B(r)g(u,v) \end{cases} \quad (2.1)$$

$$\begin{cases} u(1)=0, u'(0)=0 \\ v(1)=0, v'(0)=0 \end{cases} \quad (2.2)$$

Where

$i) A,B\in C^1(\mathbb{R}^+), A,B\geq\epsilon_1\ on\ [0,+\infty];$
$ii) f,g\in C^1(\mathbb{R}^2), uf(u,v)>0, vg(u,v)>0,$ when $u,v\neq 0, f,g\geq\epsilon_2>0\ on\ (0,+\infty], r\geq 0$ then $f(0,0)=0, g(0,0)=0$.

Lemma (2.1) [1] Let $\{r_i\}_{i=1}^k$ be zeros of an eigenfunction y_k for (1.3) corresponding to λ_k satisfying

$0 = r_0 < r_1 < r_2 < \cdots < r_{k-1} < r_k = 1.$

$i)$ Assume $\lambda > \lambda_k$, for each $1 \le i \le k$, there exist a solution z_i of

$-\left(r^{n-1}|z'|^{p-2}z'\right)' + \lambda r^{n-1}w(z)|z|^{p-2}z = 0$,(2.3)

Which has at least two zeros in (r_{i-1}, r_i).

$ii)$ Assume $\lambda < \lambda_k$, for each $1 \le i \le k$, there exist a solution $\bar{z}_i$ of (2.3) satisfying $\bar{z}_i(r)$ on $[r_{i-1}, r_i]$.

Lemma (2.2) [1] Let $M > 0$,

$w^* = max\{\psi(r) | r \in [0,1]\}$ and α satisfy $k_2 + k_1 F(\alpha) > w^* F(M)$. We define

$\delta \equiv Mp^{-\frac{1}{p}}\left(k_2 + k_1 F(\alpha) - w^* F(M)\right)^{-\frac{1}{p}} > 0$, (2.4).

Then the solution $u(r,\alpha)$ of (1.1) , (⁎) has the following properties:

$i)$ if $u(r,\alpha)$ has no zero in (r_1, r_2) and satisfies∈ $|u(r,\alpha)| \le M$ on $[r_1, r_2]$ for some $r_1, r_2 \in [0,1]$, then we have $r_2 - r_1 \le \delta$.

$ii)$ if $u(r,\alpha)$ has no zero in (x,y) for some $x, y \in [0,1]$ satisfying $y - x > 2\delta$, then $|u(r,\alpha)| > M$ for $r \in (x+\delta, y-\delta)$.

In this paper, I extend the result of [1]. I am interested in investigating nonlinear the (p,q)-Laplacian system (2.1) and (2.2).

3. MAIN RESULTS

In this section we state and prove some lemmas and then a theorem to prove the existence of positive solutions for system (2.1), (2.2).

We introduce substitution for the solution $u(r,\alpha)$ of (2.1) and (2.2) by using the generalized sine function $S_p(r)$ has been well studied by ([3], [4], [6], [7]).

The function S_p, S_q satisfies

$$\begin{cases} |S_p(r)|^p + \frac{|S'_p(r)|^p}{p-1} = 1 \\ |S_q(r)|^q + \frac{|S'_q(r)|^q}{q-1} = 1 \end{cases} ,(3.1)$$

And

$$\begin{cases} (|S'_p|^{p-2}S'_p)' + |S_p|^{p-2}S_p = 0 \\ (|S'_q|^{q-2}S'_q)' + |S_q|^{q-2}S_q = 0 \end{cases} ,(3.2)$$

In addition ,

$$\begin{cases} \pi_p \equiv 2\int_0^{(p-1)^{\frac{1}{p}}} \frac{dt}{(1-\frac{t^p}{p-1})^{\frac{1}{p}}} = \frac{2(p-1)^{\frac{1}{p}}\pi}{p\sin(\frac{\pi}{p})} \\ \pi_q \equiv 2\int_0^{(q-1)^{\frac{1}{q}}} \frac{dt}{(1-\frac{t^q}{q-1})^{\frac{1}{q}}} = \frac{2(q-1)^{\frac{1}{q}}\pi}{p\sin(\frac{\pi}{q})} \end{cases}$$

So, $S_p\left(\frac{\pi_p}{2}\right) = 1, S'_p(0) = 1,$
$S_q\left(\frac{\pi_q}{2}\right) = 1, S'_q(0) = 1,$
$S'_p\left(\frac{\pi_p}{2}\right) = 0, S'_q\left(\frac{\pi_q}{2}\right) = 0$

Now, I define phase-plane coordinates $\rho_i > 0$ and θ_i for solutions $u(r,\alpha)$, $v(r,\alpha)$ of (2.1) and (2.2) as following

$$\begin{cases} u(r,\alpha)^{p-2}u(r,\alpha) = \rho_1(r,\alpha)|S_p(\theta_1(r,\alpha))|^{p-2}S_p(\theta_1(r,\alpha)), \\ v(r,\alpha)^{q-2}v(r,\alpha) = \rho_2(r,\alpha)|S_q(\theta_2(r,\alpha))|^{q-2}S_q(\theta_2(r,\alpha)) \end{cases}$$

$$\begin{cases} r^{n-1}|u'(r,\alpha)|^{p-2}u'(r,\alpha) = \rho_1(r,\alpha)|S'_p(\theta_1(r,\alpha))|^{p-2}S'_p(\theta_1(r,\alpha)), \\ r^{n-1}|v'(r,\alpha)|^{q-2}v'(r,\alpha) = \rho_2(r,\alpha)|S'_q(\theta_2(r,\alpha))|^{q-2}S'_q(\theta_2(r,\alpha)) \end{cases}$$

With $\theta_1(0,\alpha) = \frac{\pi_p}{2}$, $\theta_2(0,\alpha) = \frac{\pi_q}{2}$. Then

$$\begin{cases} \rho_1^{\frac{p}{p-1}}(r,\alpha) = |u(r,\alpha)|^p + \frac{r^{\frac{p(n-1)}{p-1}}}{p-1}|u'(r,\alpha)|^p \\ \rho_2^{\frac{q}{q-1}}(r,\alpha) = |v(r,\alpha)|^q + \frac{r^{\frac{q(n-1)}{q-1}}}{q-1}|v'(r,\alpha)|^q \end{cases} \quad (*)$$

So, $\frac{r^{n-1}|u'|^{p-2}u'}{|u|^{p-2}u} = \frac{|S'_p|^{p-2}S'_p}{|S_p|^{p-2}S_p}$,

$\frac{r^{n-1}|v'|^{q-2}v'}{|v|^{q-2}v} = \frac{|S'_q|^{q-2}S'_q}{|S_q|^{q-2}S_q}$,

After differentiating with respect to r, we have

$$\theta'_1(r,\alpha) = \frac{r^{n-1}A(r)f(u,v)}{(p-1)|u|^{p-2}u}|S_p(\theta_1(r,\alpha))|^p + r^{\frac{1-n}{p-1}}|S'_p(\theta_1(r,\alpha))|^p, (3.3)$$

$$\frac{r^{n-1}A(r)f(u,v)}{(p-1)\rho_1(r,\alpha)}S_p(\theta_1(r,\alpha)) + r^{\frac{1-n}{p-1}}|S'_p(\theta_1(r,\alpha))|^p \equiv C(r,\alpha,\theta_1)$$

$$\frac{\rho'_1(r,\alpha)}{\rho_1(r,\alpha)} = (r^{\frac{1-n}{p-1}} - \frac{r^{n-1}A(r)fu,v)}{|u|^{p-2}u})|S_p(\theta_1(r,\alpha))|^{p-2}S_p(\theta_1(r,\alpha))S'_p(\theta_1(r,\alpha))$$

$$\theta'_2(r,\alpha) = \frac{r^{n-1}B(r)g(u,v)}{(q-1)|v|^{q-2}v}|S_q(\theta_2(r,\alpha))|^q + r^{\frac{1-n}{q-1}}|S'_q(\theta_2(r,\alpha))|^q, (3.4)$$

$$\frac{r^{n-1}B(r)g(u,v)}{(q-1)\rho_2(r,\alpha)}|S_q(\theta_2(r,\alpha))|^q + r^{\frac{1-n}{q-1}}|S'_q(\theta_2(r,\alpha))|^q \equiv D(r,\alpha,\theta_2)$$

Also we have

$$\frac{\rho'_2(r,\alpha)}{\rho_2(r,\alpha)} = (r^{\frac{1-n}{q-1}} - \frac{r^{n-1}B(r)g(u,v)}{|v|^{q-2}v}|S_q(\theta_2(r,\alpha))|^{q-2}S_q(\theta_2(r,\alpha))S'_q(\theta_2(r,\alpha))$$

The phase function for (2.1) and (2.2) with $\lambda = \lambda_k$, we conclude

$$\phi'_{1k}(r,\lambda_k) = \frac{r^{n-1}\lambda_k A(r)}{(p-1)}|S_p(\phi_{1k}(r,\lambda_k))|^p + r^{\frac{1-n}{p-1}}|S_p(\phi_{1k}(r,\lambda_k))|^p$$

$$\equiv F(r,\lambda_k,\phi_1), \quad (3.5)$$

$$\phi'_{2k}(r,\lambda_k) = \frac{r^{n-1}\lambda_k B(r)}{(q-1)}|S_q(\phi_{2k}(r,\lambda_k))|^q + r^{\frac{1-n}{q-1}}|S_q(\phi_{2k}(r,\lambda_k))|^q \equiv G(r,\lambda_k,\phi_2),$$

$$\phi_{1k}(0,\lambda_k) = \frac{\pi_p}{2}, \phi_{1k}(1,\lambda_k) = k\pi_p,$$

$$\phi_{2k}(0,\lambda_k) = \frac{\pi_q}{2}, \phi_{2k}(1,\lambda_k) = k\pi_q,$$

In the rest of the paper, we consider $|(u,v)| = |u| + |v|$.

Lemma (3.1) *i*) Suppose $\lim\ sup_{|u|\to 0}\frac{f(u,v)}{|u|^{p-2}u} < \lambda_k$, $\lim\ sup_{|v|\to 0}\frac{g(u,v)}{|v|^{q-2}v} < \lambda_k$, for $k \in \mathbb{N}$, then there exists $\alpha_* > 0$ such that $\theta_1(1,\alpha) < k\pi_p$, $\theta_2(1,\alpha) < k\pi_q$, for all $\alpha \in (0,\alpha_*)$. That is the solution $(u(r,\alpha), v(r,\alpha))$ of (2.1) and (2.2) has at most k-1 zeros in $(0,1)$ for $\alpha \in (0,\alpha_*]$.

ii) Suppose $\lim\ inf_{|u|\to 0}\frac{f(u,v)}{|u|^{p-2}u} > \lambda_k$, $\lim\ inf_{|v|\to 0}\frac{g(u,v)}{|v|^{q-2}v} > \lambda_k$, for $k \in \mathbb{N}$, then there exists $\alpha_* > 0$ such that $\theta_1(1,\alpha) > k\pi_p$, $\theta_2(1,\alpha) > k\pi_q$, when $\alpha \in (0,\alpha_*]$. That is, the solution $(u(r,\alpha), v(r,\alpha))$ of (2.1) and (2.2) has at least k-1 zeros in $(0,1)$ for $\alpha \in (0,\alpha_*]$.

Proof. *i*) The assumption implies that, there exists $\delta > 0$ and $\lambda > 0$ such that $\frac{f(u,v)}{|u|^{p-2}u} < \lambda < \lambda_k$, $\frac{g(u,v)}{|v|^{q-2}v} < \lambda < \lambda_k$ for $0 < |u| + |v| < \delta$. Since $(u,v) \equiv 0$ is a solution of (2.1) and (2.2), there exists $\alpha_* > 0$ such that $|(u(r,\alpha), v(r,\alpha))| < \delta$ for $0 < \alpha < \alpha_*$ and $r \in [0,1]$. From (3.3), (3.4) we have

$$\theta'_1(r,\alpha) < \frac{r^{n-1}\lambda_k A(r)}{(p-1)}|S_p(\theta_1(r,\alpha))|^p + r^{\frac{1-n}{p-1}}|S_p(\theta_1(r,\alpha))|^p = F(r,\lambda_k,\phi_1),$$

$$\theta'_2(r,\alpha) < \frac{r^{n-1}\lambda_k B(r)}{(q-1)}|S_q(\theta_2(r,\alpha))|^q + r^{\frac{1-n}{q-1}}|S_q(\theta_2(r,\alpha))|^q = G(r,\lambda_k,\phi_2).$$

Let u_k, v_k be the solution of (1.3) with $\lambda = \lambda_k$ and ϕ_{1k}, ϕ_{2k} be its Prüfer angle, then u_k, v_k are eigenfunctions of (1.3). Thus $\phi_{1k}(1,\lambda_k) = k\pi_p$, $\phi_{2k}(1,\lambda_k) = k\pi_q$, The comparison theorem was studied by ([8], p.30), include that $\theta_1(1,\alpha) < \phi_{1k}(1,\lambda_k)$,

$\theta_2(1,\alpha) < \phi_{2k}(1,\lambda_k)$, $0 < \alpha < \alpha_*$.

ii) By assumption, we have there exist exists $\delta > 0$ and $\lambda > 0$ such that $\frac{f(u,v)}{|u|^{p-2}u} > \lambda > \lambda_k$, $\frac{g(u,v)}{|v|^{q-2}v} > \lambda > \lambda_k$ when $0 < |u| + |v| < \delta$.
Similar to (i), we get, there exists $\alpha_* > 0$ such that

$0 < |(u(r,\alpha), v(r,\alpha))| < \delta$ for $0 < \alpha < \alpha_*$ and $r \in [0,1]$. So,

$\frac{f(u(r,\alpha),v(r,\alpha))}{|u(r,\alpha)|^{p-2}u(r,\alpha)} > \lambda_k$, $\frac{g(u(r,\alpha),v(r,\alpha))}{|v(r,\alpha)|^{q-2}v(r,\alpha)} > \lambda_k$, by (3.3) and (3.4) we get

$$\theta'_1(r,\alpha) > \frac{r^{n-1}\lambda_k A(r)}{(p-1)}|S_p(\theta_1(r,\alpha))|^p + r^{\frac{1-n}{1-p}}|S_p(\theta_1(r,\alpha))|^p = F(r,\lambda_k,\phi_1)$$

$$\theta'_2(r,\alpha) > \frac{r^{n-1}\lambda_k B(r)}{(q-1)}|S_q(\theta_2(r,\alpha))|^q + r^{\frac{1-n}{1-q}}|S_q(\theta_2(r,\alpha))|^q = G(r,\lambda_k,\phi_2).$$

Similar as in (i), we have $\theta_1(1,\alpha) > k\pi_p$, $\theta_2(1,\alpha) > k\pi_q$.

Lemma (3.2) ***i***) Assume that $\lim\ \inf_{|u|\to\infty} \frac{f(u,v)}{|u|^{p-2}u} > \lambda_k$, $\lim\ \inf_{|v|\to\infty} \frac{g(u,v)}{|v|^{q-2}v} > \lambda_k$, for $k \in \mathbb{N}$, then there exists $\alpha^* > 0$ such that the solution $(u(r,\alpha), v(r,\alpha))$ has at least k zeros in $(0,1) \times (0,1)$ for $\alpha \in [\alpha^*, \infty)$.

ii) Assume that $\lim\ \sup_{|u|\to\infty} \frac{f(u,v)}{|u|^{p-2}u} < \lambda_k$, $\lim\ \sup_{|v|\to\infty} \frac{g(u,v)}{|v|^{q-2}v} < \lambda_k$, for $k \in \mathbb{N}$, then there exists $\alpha^* > 0$ such that the solution $(u(r,\alpha), v(r,\alpha))$ has at most (k -1) zeros in $(0,1) \times (0,1)$ for $\alpha^* < \alpha$.

Proof. ***i***) by assumption, there exist $\lambda > \lambda_k$ and $M > 0$ such that $\frac{f(u,v)}{|u|^{p-2}u} > \lambda > \lambda_k$, $\frac{g(u,v)}{|v|^{q-2}v} > \lambda > \lambda_k$ when $|u| + |v| \geq M$, (3.6).

Let u_k, v_k be the k-th eigenfunction of (1.3) corresponding to λ_k and $\{r_i\}_{i=1}^{k}$ be zeros of u_k, v_k with $r_0 = 0$ and $r_k = 1$. Lemma (2.1) implies that, there exists a solution z_{1i}, z_{2i} of (2.3) having at least two zeros in (r_{i-1}, r_i). Now, fix $i \in \{1,2,\dots,k\}$, let t_1, t_2 be zeros of z_{1i}, z_{2i} satisfying $r_{i-1} < t_1 < t_2 < r_i$. By (2.4) and remark that δ tends to zero as α tends to infinity. For this i, we can choose an $\alpha_i > 0$ such that $r_i - r_{i-1} > 2\delta_i$ and

$[t_1,t_2] \subset (r_{i-1} + \delta_i,\ r_i - \delta_i)$, where α_i *and* δ_i are consistent with (2.4). Let $\alpha \geq \alpha_i$, we prove $u(r,\alpha), v(r,\alpha)$ have at least one zero in (r_{i-1}, r_i). Suppose that $u(r,\alpha), v(r,\alpha)$ have no zero in (r_{i-1}, r_i). Lemma (2.2) (***ii***) implies that $|u(r,\alpha)| > M, |v(r,\alpha)| > M$, when

$r \in (r_{i-1} + \delta_i,\ r_i - \delta_i)$. From (3.6), we have $\lambda A(r) < \frac{A(r) f(u(r,\alpha),v(r,\alpha))}{(u(r,\alpha))^{p-1}}$,

$\lambda B(r) < \frac{B(r) g(u(r,\alpha),v(r,\alpha))}{(v(r,\alpha))^{q-1}}$, for $r \in [t_1,t_2] \subset (r_{i-1} + \delta_i,\ r_i - \delta_i)$.

Then (in [5],p. 182) implies that $u(r,\alpha), v(r,\alpha)$ have at least one zero in (t_1,t_2). This leads to a contradiction. Hence $u(r,\alpha), v(r,\alpha)$ with $\alpha \geq \alpha_i$ have at least one zero in (r_{i-1}, r_i).

Set $\alpha^* = max\{\alpha_i | i = 1,2, \ldots, k\}$. If $\alpha \geq \alpha^*$, then $u(r,\alpha), v(r,\alpha)$ have at least one zero in (r_{i-1}, r_i) for each $i = 1,2, \ldots, k$. It means that $u(r,\alpha), v(r,\alpha)$ have at least k zeros in $(0,1)$ for $\alpha \in [\alpha^*, \infty)$.

ii) by assumption, there exist $\lambda < \lambda_k$ and $M > 0$ such that

$\frac{f(u,v)}{|u|^{p-2}u} < \lambda < \lambda_k$, $\frac{g(u,v)}{|v|^{q-2}v} < \lambda < \lambda_k$ when $|u| + |v| \geq M$, (3.7).

For every $\alpha > 0$, let $\phi_i(r,\alpha)$ and $\phi_{ik}(r,\alpha)$ be the Prüfer angle of the solutions of (3) with λ *and* λ_k . So,
$\phi_{1k}(1,\alpha) = k\pi_p$, $\phi_{2k}(1,\alpha) = k\pi_q$, hence by the comparison theorem,
$\phi_1(1,\alpha) = k\pi_p - \varepsilon$, $\phi_2(1,\alpha) = k\pi_q - \varepsilon, \varepsilon > 0$
and from (3.3) and (3.4) $\phi_i(r,\alpha)$ satisfying

$$\phi'_1(r,\alpha) = \frac{r^{n-1}\lambda A(r)}{(p-1)} |S_p(\phi_1(r,\alpha))|^p + r^{\frac{1-n}{p-1}} |S_p(\phi_1(r,\alpha))|^p$$

$$\equiv F(r,\alpha,\phi_1),$$

$$\phi'_2(r,\alpha) = \frac{r^{n-1}\lambda B(r)}{(q-1)} |S_q(\phi_2(r,\alpha))|^q + r^{\frac{1-n}{q-1}} |S_q(\phi_2(r,\alpha))|^q$$
$$\equiv G(r,\lambda_k,\phi_2), \quad (3.8)$$

Define:

$$R(r,\alpha) = \begin{cases} \frac{f(u(r,\alpha),v(r,\alpha))}{|u(r,\alpha)|^{p-2}u(r,\alpha)} & , |u(r,\alpha)| < M \\ \lambda & , |u(r,\alpha)| \geq M \end{cases}$$

$$T(r,\alpha) = \begin{cases} \frac{g(u(r,\alpha),v(r,\alpha))}{|v(r,\alpha)|^{q-2}v(r,\alpha)} & , |v(r,\alpha)| < M \\ \lambda & , |v(r,\alpha)| \geq M \end{cases}$$

By (3.3) and (3.4) and comparing with (3.8) there exists a sufficiently large α^*,

$\left|\frac{f(u(r,\alpha),v(r,\alpha))}{\rho_1(r,\alpha)}\right|$, $\left|\frac{g(u(r,\alpha),v(r,\alpha))}{\rho_2(r,\alpha)}\right|$ can be small for $|u(r,\alpha)| < M$, $|v(r,\alpha)| < M$ and $\alpha \geq \alpha^*$. So $\theta_1(r,\alpha), \theta_2(r,\alpha)$ are uniformly bounded for $\alpha \geq \alpha^*$ and $r \in [0,1]$. The number of zeros of $u(r,\alpha), v(r,\alpha)$ of (1.1) and (⁎) is uniformly bounded for $\alpha \geq \alpha^*$.

Also, we have $\lim_{\alpha\to\infty} \|I_{M,\alpha}\| = 0$ (3.9) when $I_{M,\alpha} = \{r \in [0,1] | |u(r,\alpha)| < M\}$, now, let $\psi_i(r,\alpha)$ be the solution of the equation
$\psi'_i(r,\alpha) = H_i((r,\alpha,\psi_i), i = 1,2,$ (3.10)
satisfying $\psi_1(0,\alpha) = \frac{\pi_p}{2}$, $\psi_2(0,\alpha) = \frac{\pi_q}{2}$ and from (3.5) with $\lambda = \lambda_k$ and (3.9) we obtain (for $\alpha \geq \alpha^*$ and $r \in [0,1]$)

$$\psi_1(r,\alpha) - \phi_1(r,\alpha) = \int_0^r (H(s,\alpha,\psi_1) - F(s,\alpha,\phi_1))ds$$

$$= \int_0^r (H(s,\alpha,\psi_1) - F(s,\alpha,\psi_1) + F(s,\alpha,\psi_1) - F(s,\alpha,\phi$$

$$\int_0^r \frac{s^{n-1}}{p-1} A(s)(R(s,\alpha) - \lambda)|S_p(\psi_1(s,\alpha))|^p ds$$

$$+ \int_0^r \frac{\partial}{\partial\phi_1} F(s,\alpha,\xi_1)(\psi_1(s,\alpha) - \phi_1(s,\alpha))ds,$$

And

$$\psi_2(r,\alpha) - \phi_2(r,\alpha) =$$
$$\int_0^r \frac{s^{n-1}}{q-1} B(s)(T(s,\alpha) - \lambda)|S_q(\psi_2(s,\alpha))|^q ds$$

$$+ \int_0^r \frac{\partial}{\partial\phi_2} G(s,\alpha,\xi_2)(\psi_2(s,\alpha) - \phi_2(s,\alpha))ds,$$

Where $\xi_i(s,\alpha)$ is between $\psi_i(s,\alpha)$ and $\phi_i(s,\alpha)$. By (3.9) , we get

$$\left|\int_0^r \frac{s^{n-1}}{p-1} A(s)(R(s,\alpha) - \lambda)|S_p(\psi_1(s,\alpha))|^p ds\right|$$

$$\leq \int_{I_{M,\alpha}} \frac{s^{n-1}}{p-1} A(s)(R(s,\alpha) - \lambda)ds < \delta$$

And also we have

$$\left|\int_0^r \frac{s^{n-1}}{q-1} B(s)(T(s,\alpha)-\lambda)\left|S_q(\psi_2(s,\alpha))\right|^q ds\right|$$

$\leq \int_{I_{M,\alpha}} \frac{s^{n-1}}{q-1} B(s)(T(s,\alpha)-\lambda)\,ds < \delta$ when $\alpha \geq \alpha^*$, $\delta > 0$. Note that $\left|\frac{\partial}{\partial \phi_1} F(s,\alpha,\xi_1)\right|$ and $\left|\frac{\partial}{\partial \phi_2} G(s,\alpha,\xi_2)\right|$ are bounded by $k_1, k_2 > 0$.

So, we have

$$|\psi_1(r,\alpha)-\phi_1(r,\alpha)| < \delta + \int_0^r k_1|\psi_1(s,\alpha)-\phi_1(s,\alpha)|ds$$

$$|\psi_2(r,\alpha)-\phi_2(r,\alpha)| < \delta + \int_0^r k_2|\psi_2(s,\alpha)-\phi_2(s,\alpha)|ds$$

If $\delta < \varepsilon e^{-k_1}, \delta < \varepsilon e^{-k_2}$, By the Gronwell inequality, we obtain

$$|\psi_1(r,\alpha)-\phi_1(r,\alpha)| < \delta e^{k_1 r} < \varepsilon,$$
$$|\psi_2(r,\alpha)-\phi_2(r,\alpha)| < \delta e^{k_2 r} < \varepsilon.$$

Hence $\psi_1(r,\alpha) < \phi_1(r,\alpha)+\varepsilon$, $\psi_2(r,\alpha) < \phi_2(r,\alpha)+\varepsilon$,

So,

$$\theta_1(r,\alpha) \leq \psi_1(r,\alpha) < \phi_1(r,\alpha)+\varepsilon = k_1\pi_p,$$

$$\theta_2(r,\alpha) \leq \psi_2(r,\alpha) < \phi_2(r,\alpha)+\varepsilon = k_2\pi_q,$$

Now, the proof is completed.

Theorem (3.3) Suppose that there exists an integer $k \in \mathbb{N}$ such that

$$\lim\, sup_{|u|\to 0} \frac{f(u,v)}{|u|^{p-2}u} < \lambda_k < \lim\, inf_{|u|\to\infty} \frac{f(u,v)}{|u|^{p-2}u}, \quad (3.11)$$

$$\lim\, sup_{|v|\to 0} \frac{g(u,v)}{|v|^{q-2}v} < \lambda_k < \lim\, inf_{|v|\to\infty} \frac{g(u,v)}{|v|^{q-2}v}, \quad (3.12)$$

Then (2.1) and (2.2) have a solution with at most k-1 zeros in (0,1).

Proof. By (3.11) and lemma (3.1) (i), there exits $\alpha_* > 0$ such that

$\theta_1(1,\alpha) < k\pi_p, \theta_2(1,\alpha) < k\pi_q$ for $\alpha \leq \alpha_*$. Lemma (3.2) (i) implies that there exits $\alpha^* > 0$ such that $\theta_1(1,\alpha) > k\pi_p, \theta_2(1,\alpha) > k\pi_q$ for $\alpha \geq \alpha^*$. Since

$\theta_1(1,\alpha) = k\pi_p, \theta_2(1,\alpha) = k\pi_q$. Similarly (3.12) can be proved. Now the proof is completed.

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