Numerical solution of Inviscid Burgers' equation by using Galerkin Finite element method

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Abstract - *In recent years, many more of the numerical methods were used to solve a wide range of mathematical, physical and engineering problems, linear and nonlinear. In this paper, the Galerkin Finite element method is used to find the numerical solution of the Inviscid Burgers' equation with initial and boundary conditions. Here, Galerkin finite element method (GFEM) is employed to approximate the solution of the Burgers equation which is one dimensional and non-linear differential equation.*

Key Words: Inviscid Burgers equation, Finite element method, Non-linear partial differential equation, Numerical techniques.

1.INTRODUCTION

The nonlinear equations are the most important phenomena across the world of Mathematics. Nonlinear phenomena have important efficiency on applied mathematics, physics, and issues related to engineering. The importance of obtaining the exact solution of nonlinear partial differential equations in physics and applied mathematics is still a big problem that needs new methods.

Inviscid Burgers' equation is essential for their role in modeling a wide array of physical systems such as traffic flow, shallow water waves, and gas dynamics [1]. The Inviscid Burger's equation also provide fundamental pedagogical examples for many important topics in nonlinear partial differential equations such as traveling waves, shock formation, similarity solutions, singular perturbation, and numerical methods for parabolic and hyperbolic equations.

In the resent years, many researchers mainly had paid attention to study the solution of nonlinear partial differential equations by using various methods. In the resent years, many researchers mainly had paid attention to studying the solution of nonlinear partial differential equations by using various methods. Among these domain decomposition method [2], homotopy perturbation method[3], variational iteration method, differential transform method[4], [5], projected differential transform method [6], Finite element methods [9] etc. are widely used. Recently researcher has solved Burgers' equation for an incompressible fluid with different viscosity by using Galerkin finite element method [7].

In this paper, Finite element method [8] has been explained. The numerical solution of Inviscid Burgers' equation has been obtained using Galerkin Finite Element method [9] with specific initial and boundary conditions.

2. MATHEMATICAL FORMULATION

Burgers' equation is a partial differential equation that was originally proposed as a simplified model of Navier-Stokes equations [10]. For a Newtonian incompressible fluid, the Navier-Stokes equations is

$$\rho\left(\frac{\partial u}{\partial t}+u\cdot\nabla u\right)=-\nabla p+\mu\nabla^{2}u+F$$

Further we assuming that, there is no external forces, neglecting pressure term and also assuming that there is no diffusive term. After doing some mathematics above equation reduce to

$$\frac{\partial u}{\partial t}+u\cdot\nabla u=0$$

In the special case of one space dimension (n=1). Then the above equation becomes

$$\frac{\partial u}{\partial t}+u\cdot\frac{\partial u}{\partial x}=0$$

We may consider this in some spatial region Ω for positive times $t\geq 0$ with the appropriate boundary conditions and initial condition.

3. MATHEMATICAL FORMULATION

We wish to numerically solve the following initial-boundary value problem corresponding to special case of Burgers' equation for an incompressible fluid:

$$\frac{\partial u}{\partial t}+u\cdot\frac{\partial u}{\partial x}=0; \qquad t\geq 0$$

Initial Condition: $u(x,0)=1-\cos x$
Boundary Condition: $u(0,t)=0=u(1,t)$
on the interval [0,1].

4. METHOD OF SOLUTION

Here we use the Galerkin finite element method, for that first we develop weak form of

$$\frac{\partial u}{\partial t}+u\cdot\frac{\partial u}{\partial x}=0$$

4.1 Weak form

We develop the weak form of equation (3) over the typical element. The first step is to multiply (3) with a weight function w, which is assumed to be differentiable once with respect to x, and then integrate the equation over the typical element domain.

$$\int_{\Omega_e} w\left[\frac{\partial u}{\partial t}+u\cdot\frac{\partial u}{\partial x}\right]dx=0$$

In the next step we distribute the differentiation among u and w equally. We get weak form as,

$$\int_{\Omega_e}\left[w\cdot\frac{\partial u}{\partial t}-\frac{1}{2}\cdot\frac{\partial u}{\partial x}\cdot u^2\right]dx=\overline{Q}$$

Where, $\overline{Q}=Q_1+Q$ and

$$Q_1=\frac{1}{2}\cdot w\cdot u^2\Big|_{x_a} \quad and \quad Q_2=-\frac{1}{2}\cdot w\cdot u^2\Big|_{x_b}$$

4.2 Finite element model

We developed Finite element models of time dependent problems, We obtain solution in the form,

$$u(x,t_s)\approx u_h^e(x,t_s)=\sum_{j=1}^{n}\left(u_j^s\right)^e\Psi_j^e(x), \quad where, s=1,2,...$$

Where, $\left(u_j^s\right)^e$ is the value of $u(x,t)$ at time $t=t_s$ and node j of the element Ω_e. Substituting $w=\Psi_i(x)$ (to obtain the i^{th} equation of the system) and equation no (6) into equation no (5), we obtain

$$\sum_{j=1}^{n}\int_{\Omega_e}\left[\left\{\Psi_i(x)\cdot\Psi_j(x)\right\}\frac{du_j}{dt}\right]dx-\frac{1}{2}\sum_{j=1}^{n}\int_{\Omega_e}\left[\left\{\frac{d\Psi_i}{dt}\cdot\Psi_j(x)\right\}u_j^{\,2}(t)\right]dx=Q_i$$

$$\sum_{j=1}^{n}M_{ij}\frac{du_j}{dt}+\sum_{j=1}^{n}A_{ij}u_j^{\,2}=Q_i$$

Where,

$$\left.\begin{aligned}M_{ij}&=\int_{\Omega_e}\Psi_i(x)\Psi_j(x)\,dx\\ A_{ij}&=-\frac{1}{2}\int_{\Omega_e}\frac{d\Psi_i}{dx}\Psi_j(x)\,dx\end{aligned}\right\} \qquad (7)$$

4.3 Time discretization

Equation (7) includes nodal unknowns and their time derivatives. Therefore it is not a set of algebraic equations, but instead a set of ODEs, There for we discretize the time derivatives of the nodal unknowns. For that we use forward difference scheme and after doing some mathematics, we get follows to solve for the unknowns at time level $s+1$

$$\{u\}_{s+1}=[M]^{-1}\left[[M]\{u\}_s+\Delta t\left(Q-[A]\{u^2\}_s\right)\right]$$

At $x=0$, nodal unknown vectors $\{u\}_0$, can be obtained from the given initial conditions, By using the appropriate boundary and initial condition for different time we obtain solution for equation (8), which is shown in Table 1.

Table -1: Computational results

	$t=0.1$	$t=0.3$	$t=0.5$	$t=0.7$	$t=1.0$
$x=0$	0	0	0	0	0
$x=0.1$	0.4433	0.4084	0.3732	0.3393	0.2928
$x=0.2$	1.3273	1.1733	1.0447	0.9351	0.7978
$x=0.3$	1.9060	1.7474	1.6019	1.4696	1.2935
$x=0.4$	1.6516	1.6305	1.5919	1.5415	1.4523
$x=0.5$	0.7981	0.9336	1.0357	1.1094	1.1767
$x=0.6$	0.1302	0.2977	0.4463	0.5754	0.7338
$x=0.7$	0.2620	0.3071	0.3644	0.4284	0.5281
$x=0.8$	1.0708	0.9431	0.8441	0.7714	0.7039
$x=0.9$	1.7441	1.4603	1.2334	1.0530	0.8523
$x=1.0$	0	0	0	0	0

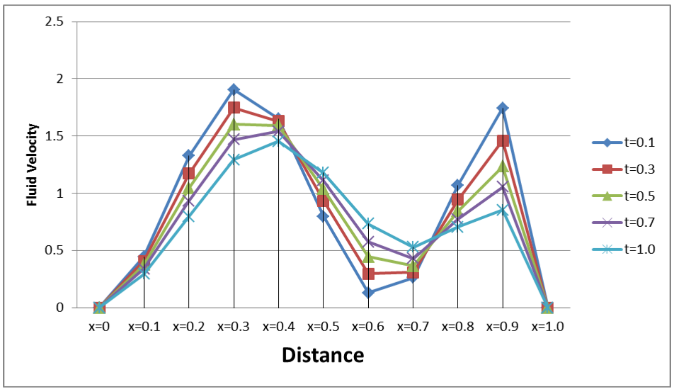

Chart -1: Graphical representation of solution for different time interval

The Inviscid Burgers' equation is non-linear partial differential equation and hence it is bit difficult to find its analytic solution. The numerical solution of Inviscid Burgers' equation is obtained for particular values of the parameters by GFEM for some countable meshes, which we calculate easily, to get much effective result we have to derive result for more number of mashes, for that we using MATLAB and its graphical representation is given in below diagrams for same boundary and initial conditions.

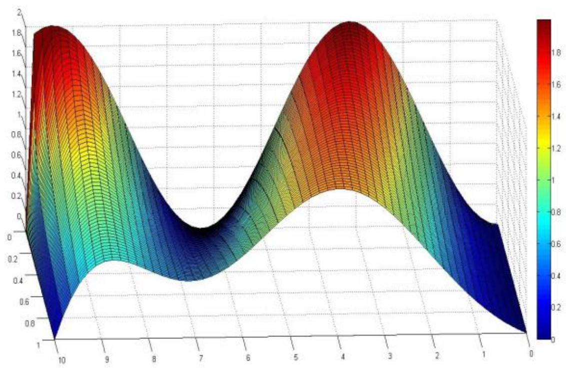

Fig -1: Solution of Inviscid Burgers equation by using MATLAB

5. CONCLUSIONS

A specific problem of one-dimensional flow in unsaturated porous media under certain assumptions leads to Burger equation, we discussed it's special case; Inviscid Burgers' equation and its numerical solution is obtained by the Galerkin Finite element method. Easy in use and computationally cost-effectiveness has made the present method an efficient alternative to some rival methods in solving the problems modeled by the non-linear problems. Also it is convenient and quite accurate to systems of partial differential equations. The obtained results show that the method is also a promising method to solve other nonlinear partial differential equations.

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