OBSERVATIONS ON TERNARY QUADRATIC EQUATION $z^2 = 82x^2 + y^2$

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Abstract-*The Quadratic Diophantine Equation with three unknowns $z^2 = 82x^2 + y^2$ is analyzed for its non-zero distinct integer solutions. A few interesting relations among the solutions and some special polygonal, Octahedral and Four dimensional figurate numbers are presented.*

Keywords

Quadratic equation with three unknowns, Integral solutions, Polygonal Numbers, Four Dimensional Figurate Numbers, Octahedral number.

1. Introduction

Ternary quadratic equations are rich in variety. For an extensive review of sizable literature and various problems one may refer [1-7]. In [8], the ternary quadratic Diophantine equation of the form $kxy + m(x+y) = z^2$ has been studied for its non-trivial integral solutions. In [9,10], A special Pythagorean triangle problem have been discussed for its integral solutions. In [11], two parametric non-trivial integral solutions of the ternary quadratic homogeneous Diophantine equation $X^2 + PXY + Y^2 = Z^2$, where P is a non-zero constant have been presented. In [12], the ternary quadratic homogeneous equation $k\alpha(x^2 + y^2) + bxy = 4k\alpha^2 z^2$, (k, α ,b $\neq 0$) has been studied for its non-trivial integral solutions. In [13], the ternary quadratic Diophantine equation of the form $(x-y)(x-z) + y^2 = 0$, is analyzed for its integral solutions at different angles and their parametric representations are obtained.

In this communication, we consider yet another interesting ternary quadratic equation $z^2 = 82x^2 + y^2$ and obtain different patterns of non-trivial integral solutions. Also, a few interesting relations among the solutions and special polygonal, Octahedral and Four dimensional numbers are presented.

2. Notations

Denoting the ranks of some special polygonal, Octahedral and Four dimensional figurate numbers.

$T_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$ = Polygonal Number with rank n and sides m .

$O_n = \frac{(2n^3 + n)}{3}$ = Octahedral Number of rank n.

$4DF_n = \frac{n^2(n^2-1)}{12}$ = Four dimensional figurate Number whose generating polygon is a square.

3.Method of Analysis

The ternary quadratic Diophantine equation to be solved for its non-zero integral solution is

$$z^2 = 82x^2 + y^2. \quad (1)$$

Assuming $z = z(a,b) = a^2 + 82b^2$. (2)

where a and b are non-zero integers.

Pattern : 1

Equation (1) can be written as

$$z^2 - 82x^2 = y^2 \quad (3)$$

Assuming $y = y(a,b) = a^2 - 82b^2$ (4)

we get

$$\left(a^2 + 82b^2\right)^2 = \left(82x^2 + y^2\right)$$

Using the factorization method, we have

$$z + \sqrt{82}x = \left(a + \sqrt{82}b\right)^2 \quad (5)$$

$$z - \sqrt{82}x = \left(a - \sqrt{82}b\right)^2 \quad (6)$$

Comparing rational and irrational factors,

$$z = z(a,b) = a^2 + 82b^2$$

$$x = x(a,b) = 2ab$$

The corresponding non-zero distinct integer solutions are

$$x = x(a,b) = 2ab$$

$$y = y(a,b) = a^2 - 82b^2$$

$$z = z(a,b) = a^2 + 82b^2$$

Observations

1. If a = b and a is even, then z is divisible by 2.
2. For all the values of a and b, x + y + z and x – y + z is divisible by 2.
3. If a is even and b is odd, then z is divisible by 2.
4. $y(a,2) - x(a,2) + z(a,2) - 2T_{8,a} \equiv 0 \pmod 4$
5. $y(a,1) + z(a,1) - 4T_{3,a} \equiv 0 \pmod 2$
6. $x(a,3) + y(a,3) + z(a,2) + 4T_{7,a} \equiv 0 \pmod{12}$
7. $x(a,1).y(a,1) + 49O_n \equiv 0 \pmod{330}$.
8. Each of the following expressions
   (i) 2x(a,a) + y(a,a) + z(a,a)

(ii) $3x(a,1) - T_{14,a} + T_{26,a}$

represents a Nasty number.

Pattern 2 :

Equation (1) can be written as

$$82x^2 + y^2 = z^2 * 1 \tag{7}$$

Assuming $z = z(a,b) = a^2 + 82b^2$.

and write

$$1 = \frac{(9 + i8\sqrt{82})(9 - i8\sqrt{82})}{5329} \tag{8}$$

Using factorization method, equation (7) can be written as

$$(y + i\sqrt{82x})(y - i\sqrt{82x}) = \left[\frac{9 + i8\sqrt{82}}{73}\right]\left[\frac{9 - i8\sqrt{82}}{73}\right]\left[(a + i\sqrt{82}b)^2 (a - i8\sqrt{82}b)^2\right]$$

We get

$$(y + i\sqrt{82}x) = \frac{1}{73}(9 + i8\sqrt{82})(a + i\sqrt{82}b)^2 \tag{9}$$

$$(y + i\sqrt{82}x) = \frac{1}{73}(9 + i8\sqrt{82})(a + i\sqrt{82}b)^2 \tag{10}$$

Comparing real and imaginary parts, we get

$$x = \frac{1}{73}[8a^2 - 656b^2 + 18ab]$$

$$y = \frac{1}{73}[9a^2 - 738b^2 - 1312ab]$$

Since our interest is on finding integer solutions, we have choose a and b suitably so that x, y and z are integers.

Let us take a = 73A and b = 73B

The integer solutions are

$$x = 584A^2 - 47888B^2 + 1314AB$$

$$y = 657A^2 - 53874B^2 - 95776AB$$

$$z = 5329A^2 + 436978B^2$$

Observations

1. If A = B, then x + y + z is divisible by 2.
2. $y(A,1) - x(A,1) - 13870\, T_{18,A} + 5986 \equiv 0 (\mathrm{mod}\ 73).$
3. $x(A,1) + z(A,1) + 109573\, T_{12,A} + 47888 \equiv 0 (\mathrm{mod}\ 73).$

Pattern 3:

Equation (1) can be written as

$$z^2 - y^2 = 82x^2$$

and we get

$$(z+y)(z-y) = 82x.x \tag{11}$$

Case 1:

Equation (11) can be written as

$$\frac{z+y}{82x} = \frac{x}{z-y} = \frac{P}{Q} \tag{12}$$

From equation (12), we get two equations

$$-82Px + Qy + Qz = 0$$

$$Qx + Py - P z = 0$$

Applying cross ratio method, we get the integer solutions are

$$x = x(P,Q) = -2PQ$$

$$y = y(P,Q) = Q^2 - 82P^2$$

$$z = z(P,Q) = -Q^2 - 82P^2$$

Observations:

1. $y(P,3) + z(P,3) + 41T_{10,P} \equiv 0 (\mathrm{mod}\ 123).$
2. $x(P,1) - z(P,1) - T_{8,P} - 1 \equiv 0 (\mathrm{mod}\ 79).$
3. For all the values of P and Q, x + y + z is divisible by 2.
4. Each of the following expressions
   (iii) y(P,P) - 2x(P,P) - z(P,P)
   (iv) $T_{24,P} - T_{12,P} - 3x(P,1).$

represents a Nasty number.

Case 2 :

Equation (11) can be written as

$$\frac{z+y}{x}=\frac{82x}{z-y}=\frac{P}{Q} \tag{13}$$

From equation (13), we have two equations

$$-Px + Qy + Qz = 0$$

$$82Qx + Py - Pz = 0$$

Applying cross ratio method, we get the integer solutions are

$$x = x(P,Q) = -2QP$$

$$y = y(P,Q) = 82Q^2 - P^2$$

$$z = z(P,Q) = -P^2 - 82Q^2$$

Observations

1. $x(1,Q)+y(1,Q)+z(1,Q)-2T_{6,Q}\equiv 0 \pmod 2$.
2. $y(2,Q).z(2,Q)+806884\ DF_{6,Q}\equiv 0 \pmod 4$.
3. $4T_{14,P}-10x(P,1)$ represents a Nasty Number.

Pattern 4:

Equation (1) can be written as

$$z^2 - 82x^2 = y^2 * 1 \tag{14}$$

and write

$$1=\left(\sqrt{82}+9\right)\left(\sqrt{82}-9\right)$$

Assuming $y = y(a,b) = a^2 - 82b^2$ (15)

we have

$$z^2-82x^2=\left(a^2-82b^2\right)^2\left[\left(\sqrt{82}+9\right)\left(\sqrt{82}-9\right)\right]$$

Using factorization method, we get

$$z+\sqrt{82}x=\left(a+\sqrt{82}b\right)^2\left[\left(\sqrt{82}+9\right)\right] \tag{16}$$

$$z+\sqrt{82}x=\left(a+\sqrt{82}b\right)^2\left[\left(\sqrt{82}+9\right)\right] \tag{17}$$

From equation (16), we get

$$z = 9a^2 + 738b^2 + 164ab$$

$$x = a^2 + 82b^2 + 18ab$$

Thus, the corresponding non-zero distinct integer solutions are

$$x = x(a,b) = a^2 + 82b^2 + 18ab$$

$$y = y(a,b) = a^2 - 82b^2$$

$$z = 9a^2 + 738b^2 + 164ab$$

Observations

1. $3x(a,a) + 3y(a,a) - 4T_{7,a} \equiv 0 \pmod 2$.
2. $x(a,1) + y(a,1) + z(a,1) + 52T_{11,a} \equiv 0 \pmod 5$.
3. $9x(2,b) - z(2,b) - T_{12,b} \equiv 0 \pmod 5$.

4. CONCLUSION

one may search for other patterns of non-zero integer solutions and relations among the solutions.

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