Deriving one dimensional shallow water equations from mass and momentum balance laws

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Abstract - *The Short Wave Equations (SWE) is a system of hyperbolic/parabolic Partial Differential Equations (PDE) that governing fluid flow in the oceans, coastal regions, estuaries, rivers and channels. In shallow water, flows there would be a relatively smaller vertical flow compared to horizontal one and in order to get rid of the vertical parameter, we can average over the depth. The SWE can be used to predict tides, storm surge levels and coastline changes like hurricanes and ocean currents. SWE also arise in atmospheric flows and debris flows. The SWE are derived from the Navier-Stokes equations, which describe the motion of fluids. The Navier-Stokes equations are themselves derived from the equations for conservation of mass and linear momentum.*

Keywords: *continuum mechanics, hydraulic jump, Navier-Stokes, Hydraulics.*

1. Introduction

Continuum mechanics is a branch of mechanics in which we study and analyze the kinematics and mechanical behavior of materials [1]. Continuum mechanics are a continuous mass that deals with the physical properties of solids and liquids. This technique could be used in many fields such as simulation of corrosion of additively manufactured parts [2], simulation of precision finishing processes [3], vibration [4], heat transfer [5], or fluid mechanics [6]. Physical properties are independent of the coordinates they take place. In this paper we will focus conservation equations on fluids and the shallow water equations [7]. The shallow water equations explains the behavior of a thin layer of fluid with a constant density that has boundary conditions from below by the bed of the flow and from above by a free surface of water. There are so many different features inside of these flows due to the fact that their behavior is based on lots of conservation laws [7].

We consider this type of flow with a simple vertical structure and we assume that fluid system is the flow of a thin layer of water over terrain, which varies in elevation. Friction is ignored and the flow velocity is assumed to be uniform with elevation [7]. Besides, the slope of terrain is assumed much less than unity, like the slope of the fluid surface. These assumptions allow the vertical pressure profile of the fluid to be determined by the hydrostatic equation [8, 9]. Although the derived shallow water equations are idealized, there are still some common essential characteristics with more complex flows such as Tsunami, deep in ocean or near shore. Near shore, a more complicated model is required [8]. Fig. 1 illustrates a schematic view of shallow water flow [1].

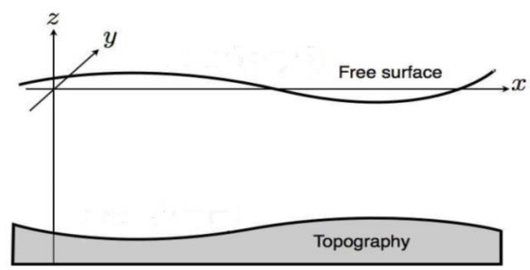

Fig.1, schematic view of a shallow water flow [1].

Hydraulic jump occurs whenever we have a flow changes from supercritical to subcritical. In phenomenon the water surface rises suddenly and we have surface rollers while an intense mixing occurs and air is entrained followed by a large amount of energy dissipation [10]. In other words, a hydraulic jump happens when a supercritical upstream meets by a subcritical downstream flow [10]. There are also artificial hydraulic jumps, which are created by devices like sluice gates. Totally, a hydraulic jump can be used as an energy dissipater, chemical mixer or to act as an aeration device [11].

Since we have unknown loss of energy in hydraulic jumps, one should conservation of momentum to derive jump equations [10]. To develop this equation, they generally consider a situation with or without loss of energy between upstream and downstream. Besides, that situation may come with or without some obstacles which may cause a drag force of P_f [11]. Fig.2 shows a schematic hydraulic jump in a flow.

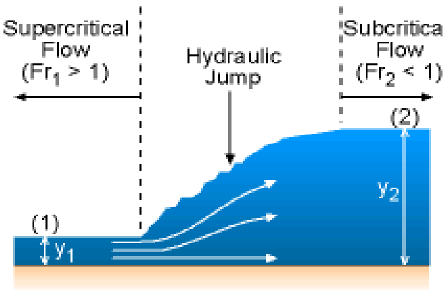

Fig.2, a schematic view of hydraulic jump [10].

2. Derivation of Navier-Stokes equations for shallow water equations

To have the final equation for shallow water, there are four steps which are (a) deriving the Navier-Stokes equations using the conservation laws, (b) ensemble the Navier-Stokes equations to account for the turbulent nature of ocean flow, (c) applying Navier-Stokes equations boundary conditions and (d) integrate the Navier-Stokes equations over depth applying the boundary conditions [9, 12-14]. In our derivation, we follow the procedure in [15]. Consider mass balance over a control volume Ω. Then:

$$\frac{d}{dt}\int_\Omega \rho dV = -\int_{\partial\Omega}(\rho\mathbf{v}).\mathbf{n}\, dA \tag{3.1}$$

Time rate of change of total mass in Ω Net mass flux across boundary of Ω

Where ρ is the fluid density (kg/m^3), $\mathbf{v}$ = (u, v, w) is the fluid velocity (m/s) and $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$. Applying Gauss's theorem gives:

$$\frac{d}{dt}\int_\Omega \rho dV = -\int_\Omega \nabla.(\rho\mathbf{v})\, dV \tag{3.2}$$

Assuming that ρ is smooth, we can apply the *Leibnitz integral* rule:

$$\int_\Omega \left[\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{v})\right] dV = 0 \tag{3.3}$$

Since Ω is arbitrary, then:

$$\left[\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{v})\right] = 0 \tag{3.4}$$

Next, consider linear momentum balance over a control volume Ω, then:

$$\frac{d}{dt}\int_\Omega \rho dV = -\int_{\partial\Omega}(\rho\mathbf{v})\mathbf{v}.\mathbf{n}\, dA + \int_\Omega \rho\mathbf{b}\, dV + \int_{\partial\Omega}\mathbf{Tn}\, dA \tag{3.5}$$

Time rate of change of total momentum in Ω Net momentum flux over boundary of Ω Body forces acting on Ω External contact forces acting on $\partial\Omega$

[illegible] on the fluid by unit mass (N/kg), and $\mathbf{T}$ is the Cauchy stress tensor (N/m^2). Furthers details on existence proof are provided [illegible] applying Gauss's Theorem and rearranging gives:

$$\frac{d}{dt}\int_\Omega \rho dV - \int_\Omega \nabla.(\rho\mathbf{vv})dV - \int_\Omega \rho\mathbf{b}\, dV - \int_\Omega \mathbf{\nabla}.\mathbf{T}\, dV = 0 \tag{3.6}$$

By assuming $\rho\mathbf{v}$ as a smooth parameter and applying the *Leibnitz integral rule* again:

$$\int_\Omega \left[\frac{\partial}{\partial t}(\rho\mathbf{v}) + \boldsymbol{\nabla}.(\rho\mathbf{v}\mathbf{v}) - \rho\mathbf{b} - \nabla.\mathbf{T}\right] dV = 0 \tag{3.7}$$

As Ω is random,

$$\left[\frac{\partial}{\partial t}(\rho\mathbf{v}) + \boldsymbol{\nabla}.(\rho\mathbf{v}\mathbf{v}) - \rho\mathbf{b} - \nabla.\mathbf{T}\right] = 0 \tag{3.8}$$

Combining the conservation of mass and linear momentum equations in their differential forms, we have:

$$\frac{\partial\rho}{\partial t} + \nabla.(\rho\mathbf{v}) = 0 \tag{3.9}$$

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla.(\rho\mathbf{v}\mathbf{v}) = \rho\mathbf{b} + \nabla.\mathbf{T} \tag{3.10}$$

In order to derive the Navier-Stokes equations from equations 3.9 and 3.10, we need to have some simplification about the fluid density, body force and stress tensor, ρ, **b** and **T**, respectively. The density of water ρ is independent of the pressure P because the water is incompressible but it does not definitely imply that the density is constant. Although in ocean modeling ρ is a function of salinity and temperature, in this case we assumed that salinity and temperature are constant throughout the sample and we can use ρ as a fixed parameter [15]. Therefore, we can simplify the equations:

$$\nabla.\mathbf{v} = 0 \tag{3.11}$$

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla.(\rho\mathbf{v}\mathbf{v}) = \rho\mathbf{b} + \nabla.\mathbf{T} \tag{3.12}$$

The form of **T** will be affected since sea water acts as a Newtonian fluid [15]. We know that gravity is one body force, so:

$$\rho\mathbf{b} = \rho\mathbf{g} + \rho\mathbf{b}_{others} \tag{3.13}$$

Where **g** is gravitational acceleration (m/s^2) and $\mathbf{b}_{others}$ are other body forces (N/kg) that we will neglect for now. For a Newtonian fluid:

$$\mathbf{T} = -\mathrm{P}\mathbf{I} + \overline{\mathbf{T}} \tag{3.14}$$

Our final Navier-Stokes equations based on pressure P (Pa) and matrix of stress terms $\overline{\mathbf{T}}$ are as 3.15 and 3.16:

$$\nabla.\mathbf{v} = 0 \tag{3.15}$$

$$\frac{\partial}{\partial t}(\rho\mathbf{v}) + \nabla.(\rho\mathbf{vv}) = -\nabla\mathrm{P} + \rho\mathbf{g} + \nabla.\overline{\mathbf{T}} \quad (3.16)$$

In addition, we have:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad (3.17)$$

$$\frac{\partial(\rho u)}{\partial t} + \frac{\partial(\rho u^2)}{\partial x} + \frac{\partial(\rho uv)}{\partial y} + \frac{\partial(\rho uw)}{\partial z} = \frac{\partial(\tau_{xx}-P)}{\partial x} + \frac{\partial(\tau_{xy})}{\partial y} + \frac{\partial(\tau_{xz})}{\partial z} \quad (3.18)$$

$$\frac{\partial(\rho v)}{\partial t} + \frac{\partial(\rho uv)}{\partial x} + \frac{\partial(\rho v^2)}{\partial y} + \frac{\partial(\rho vw)}{\partial z} = \frac{\partial(\tau_{xy})}{\partial x} + \frac{\partial(\tau_{yy}-P)}{\partial y} + \frac{\partial(\tau_{yz})}{\partial z} \quad (3.19)$$

$$\frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho uw)}{\partial x} + \frac{\partial(\rho vw)}{\partial y} + \frac{\partial(\rho w^2)}{\partial z} = -\rho g + \frac{\partial(\tau_{xz})}{\partial x} + \frac{\partial(\tau_{yz})}{\partial y} + \frac{\partial(\tau_{zz}-P)}{\partial z} \quad (3.20)$$

For boundary conditions in Fig.3, we can write:

A) $\xi = \xi(t, x, y)$, the relative elevation (m) of the free surface from geoid.

B) $b = b(x, y)$, the bathymetry (m), moves positive downward from geoid.

C) $H = H(t, x, y)$, the total distance (m) of the water column from top to bottom. We know $H = \xi + b$ [15].

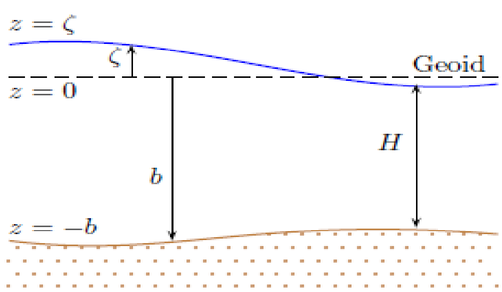

Fig.3, boundary conditions in a shallow water model [15].

We describe the boundary conditions as below [1]:

A) At the bottom of flow where $z = -b$, there is no normal flow no while slipping happening that results in $u = v = 0$. We will have:

$$u\frac{\partial b}{\partial x} + v\frac{\partial b}{\partial y} + w = 0 \quad (3.21)$$

Shear stress at the bottom in x and y directions would be:

$$\tau_{bx} = \tau_{xx}\frac{\partial b}{\partial x} + \tau_{xy}\frac{\partial b}{\partial y} + \tau_{xz} \quad (3.22)$$

$$\tau_{by} = \tau_{xy}\frac{\partial b}{\partial x} + \tau_{yy}\frac{\partial b}{\partial y} + \tau_{yz} \quad (3.23)$$

B) At the surface of flow as a free surface where $z = \xi$, we have P=0 with no normal flow that leads us to 1.24:

$$\frac{\partial \xi}{\partial t} + u\frac{\partial \xi}{\partial x} + v\frac{\partial \xi}{\partial y} - w = 0 \quad (3.24)$$

Shear stress at the surface in x and y directions would be:

$$\tau_{sx} = -\tau_{xx}\frac{\partial \xi}{\partial x} - \tau_{xy}\frac{\partial \xi}{\partial y} + \tau_{xz} \quad (3.25)$$

$$\tau_{sy} = -\tau_{xy}\frac{\partial \xi}{\partial x} - \tau_{yy}\frac{\partial \xi}{\partial y} + \tau_{yz} \quad (3.26)$$

First, we check the normal velocity related the momentum equation and then we integrate the previous equations throughout depth. With a good approximation, except pressure and gravity all of the other terms are very small and negligible which leads to a shorter z-momentum equation. The z-momentum could be derived as 3.27 [1]:

$$\frac{\partial \mathrm{P}}{\partial z} = \rho g \quad (3.27)$$

3.27 means that:

$$P = \rho g(\xi - z) \quad (3.28)$$

Equation 3.28 is called *hydrostatic distribution*. Then in x and y directions we have:

$$\frac{\partial p}{\partial x} = \rho g\frac{\partial \xi}{\partial x} \quad (3.29)$$

$$\frac{\partial p}{\partial y} = \rho g\frac{\partial \xi}{\partial y} \quad (3.30)$$

By integrating the continuity equation $\nabla.\mathbf{v} = 0$ over the boundary conditions, $z = -b$ to $z = \xi$, and since both b and ξ are t, x and y dependent, we apply *Leibnitz integral* rule as 3.31 [1]:

$$0 = \int_{-b}^{\xi} \nabla.\mathbf{v}\, dz \quad (3.31)$$

$$= \int_{-b}^{\xi}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) dz + w\big|_{z=\xi} - w\big|_{z=-b}$$

$$= \frac{\partial}{\partial x}\int_{-b}^{\xi} u\, dz + \frac{\partial}{\partial y}\int_{-b}^{\xi} v\, dz - \left(u\Big|_{z=\xi}\frac{\partial \xi}{\partial x} + u\Big|_{z=-b}\frac{\partial b}{\partial x}\right) - \left(v\Big|_{z=\xi}\frac{\partial \xi}{\partial y} + v\Big|_{z=-b}\frac{\partial b}{\partial y}\right)$$

$$+w|_{z=\xi} - w|_{z=-b}$$

Depth-averaged velocities in different directions are described as 3.32 and 3.33:

$$\bar{u} = \frac{1}{H}\int_{-b}^{\xi} u\,dz \quad (3.32)$$

$$\bar{v} = \frac{1}{H}\int_{-b}^{\xi} v\,dz \quad (3.33)$$

Combining depth-averaged velocities in different directions and using the boundary conditions to eliminate the boundary factors, we will derive depth-averaged continuity equation as 3.34:

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) + \frac{\partial}{\partial y}(H\bar{v}) = 0 \quad (3.34)$$

Integrating left hand side of both x and y momentum equations over depth, for both directions we will have:

$$\int_{-b}^{\xi}\left[\frac{\partial}{\partial t}u + \frac{\partial}{\partial x}(u^2) + \frac{\partial}{\partial y}(uv) + \frac{\partial}{\partial z}(uw)\right]dz \quad (3.35)$$

$$= \frac{\partial}{\partial t}(H\bar{u}) + \frac{\partial}{\partial x}(H\bar{u}^2) + \frac{\partial}{\partial y}(H\bar{u}\bar{v}) + \left\{\frac{Differential}{Advection\ Terms}\right\}$$

$$\int_{-b}^{\xi}\left[\frac{\partial}{\partial t}v + \frac{\partial}{\partial x}(uv) + \frac{\partial}{\partial y}(v^2) + \frac{\partial}{\partial z}(vw)\right]dz \quad (3.36)$$

$$= \frac{\partial}{\partial t}(H\bar{v}) + \frac{\partial}{\partial x}(H\bar{u}\bar{v}) + \frac{\partial}{\partial y}(H\bar{v}^2) + \left\{\frac{Differential}{Advection\ Terms}\right\}$$

Since the average of the product of two functions is not equal to the product of the averages, we use the differential advection terms and integrating over depth gives 3.37:

$$\begin{cases} -\rho g H\frac{\partial\xi}{\partial x} + \tau_{sx} - \tau_{bx} + \frac{\partial}{\partial x}\int_{-b}^{\xi}\tau_{xx} + \frac{\partial}{\partial y}\int_{-b}^{\xi}\tau_{xy} \\ -\rho g H\frac{\partial\xi}{\partial y} + \tau_{sy} - \tau_{by} + \frac{\partial}{\partial x}\int_{-b}^{\xi}\tau_{xy} + \frac{\partial}{\partial y}\int_{-b}^{\xi}\tau_{yy} \end{cases} \quad (3.37)$$

The 2D (nonlinear) conservative SWE in x and y directions can be derived by combining the left and right hand sides of the depth-integrated x and y momentum equations, 3.39 and 3.40, with the depth-integrated continuity equation, 3.34:

$$\frac{\partial H}{\partial t} + \frac{\partial}{\partial x}(H\bar{u}) + \frac{\partial}{\partial y}(H\bar{v}) = 0 \quad (3.38)$$

$$\frac{\partial}{\partial t}(H\bar{u}) + \frac{\partial}{\partial x}(H\bar{u}^2) + \frac{\partial}{\partial y}(H\bar{u}\bar{v}) = -gH\frac{\partial\xi}{\partial x} + \frac{1}{\rho}[\tau_{sx} - \tau_{bx} + F_x] \quad (3.39)$$

$$\frac{\partial}{\partial t}(H\bar{v}) + \frac{\partial}{\partial x}(H\bar{u}\bar{v}) + \frac{\partial}{\partial y}(H\bar{v}^2) = -gH\frac{\partial\xi}{\partial y} + \frac{1}{\rho}[\tau_{sy} - \tau_{by} + F_y] \quad (3.40)$$

By a case study, determination of the surface stress, bottom friction, F_x and F_y have to be done [1].

3. Hydraulic Jumps

Consider a flow in an open channel with a bump in its path, as in Fig.4. According to if the flow is supercritical or subcritical and how much is the height of the bump; the free surface acts completely different [10]. By continuity and momentum balance equations, sections 1 and 2 in Fig.4 for a flow can be related by:

$$V_1 y_1 = V_2 y_2 \quad (4.1)$$

In addition:

$$\frac{V_1^2}{2g} + y_1 = \frac{V_2^2}{2g} + y_2 + \Delta h \quad (4.2)$$

Solving 4.1, 4.2 for V_2 and the depth of y_2 over the bump gives a cubic polynomial equation:

$$y_2^3 - E_2 y_2^2 + \frac{V_1^2 y_1^2}{2g} = 0 \quad (4.3)$$

Where:

$$E_2 = \frac{V_1^2}{2g} + y_1 - \Delta h \quad (4.4)$$

Considering Δh is not too large, equation 4.4 would have two positive and one negative solution. Located on the upper or the lower leg of the energy curve in Fig.5, occurrence of condition 1 can affect the behavior of the flow. The difference between E_2 and E_1, the specific energy and the approach energy, is exactly equal to Δh. As it is shown in Fig.5, there will be a decrease and an increase in the water level over the bump for a subcritical regime with Fr < 1.0 and a supercritical regime with Fr > 1.0, respectively [11].

The flow at the crest will be exactly in a critical mode if we considering $\Delta h_{max} = E_1 - E_c$ as the bump height but for a bump height as $\Delta h > \Delta h_{max}$, there are no solid solutions to equation (4.4). The reason is a very big bump will almost block the flow path and will cause some frictional effects which results in a discontinuous solution as a hydraulic jump [11].

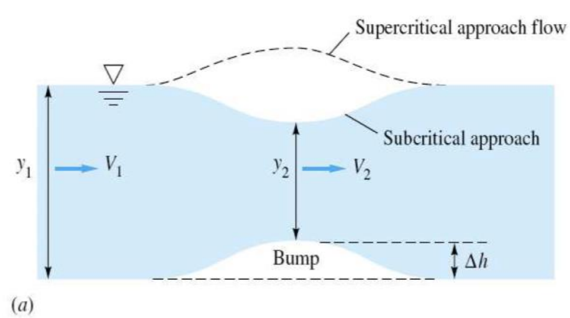

Fig.4, Frictionless two-dimensional flow over a bump [10].

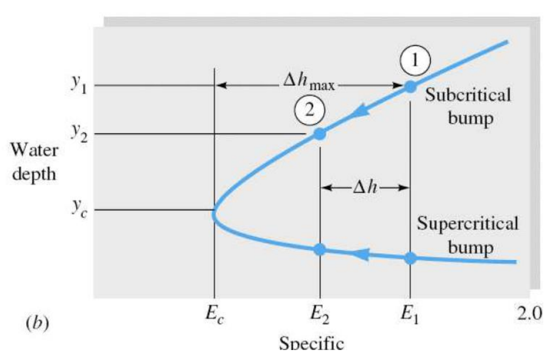

Fig.5, specific energy plots showing bump size and water depths [11].

Fluid behavior over bump will be reverse if we have a depression ($\Delta h < 0$) and it means that for a supercritical regime there would be a drop in water level while for a subcritical regime there would be a rise in the water level. In this case there is no critical flow because point 2 will be Δh to the right of point 1 [11].

Fig.6 shows there is a fast transition from supercritical to subcritical flow while the flow passes the hydraulic jump. The downstream flow is slow and deep, unlike the upstream flow which is fast and shallow [10]. The hydraulic jump can be used as a very effective energy dissipater and can be used in stilling-basin and spillway applications because it has an extremely turbulent and agitated nature. Fig.7 shows hydraulic jump at the bed of a river [11].

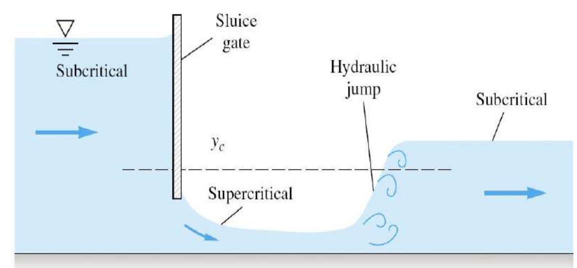

Fig.6, flow under a sluice gate accelerates from subcritical to critical to supercritical flow and then jumps back to subcritical flow [10].

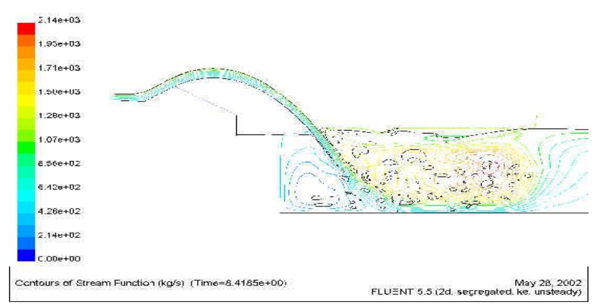

Fig.7, naturally occurring hydraulic jump formed at the bottom of a river [11, 16].

The upstream Froude number $Fr = V_1 . {}^{-2}\sqrt{g y_1}$ is the primary affecting factor on hydraulic jump performance. The secondary effect is by Reynolds number and channel geometry for real flows. Fig.8 shows the outline ranges of operation [11].

Fr < 1.0:	Violating second law of thermodynamics so jump is impossible,
Fr = 1.0 to 1.7:	*Undular jump* about $4y_2$ long with less than 5% dissipation,
Fr = 1.7 to 2.5:	*Weak jump* with a smooth rise in surface and 5% to 15% dissipation,
Fr = 2.5 to 4.5:	*Oscillating jump* with a large wave created by irregular pulsation and 15% to 45% dissipation,
Fr = 4.5 to 9.0:	Best design range because of well-balanced steady jump with 45% to 70% dissipation,
Fr > 9.0:	Rough and strong jump with a good performance and 70% to 85% dissipation.

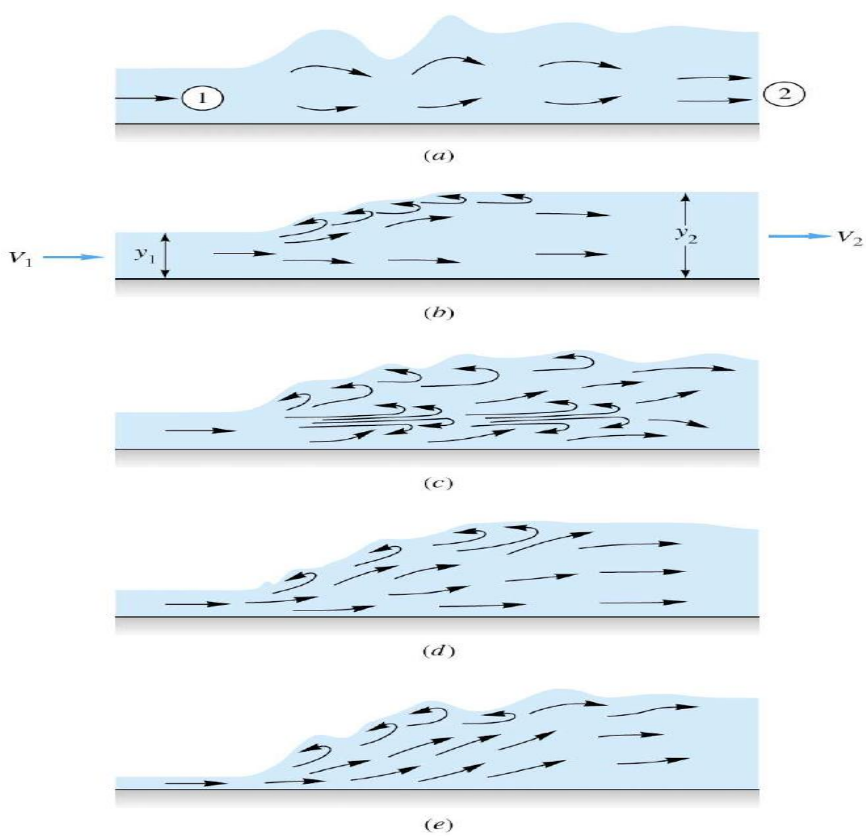

Fig.8, classification of hydraulic jumps: *(a) Fr* = 1.0 to 1.7: undular jumps; *(b) Fr* = 1.7 to 2.5: weak jumps; *(c) Fr* = 2.5 to 4.5: collating jumps; *(d) Fr* = 4.5 to 9.0: steady jumps; *(e) Fr* > 9.0: strong jump [11].

Water-weight components along the flow can affect the jump, which occurs on a steep channel slope. Regarding the classic theory we hypothesize that jump occurs on a horizontal bottom, due to the fact that this effect is small [11]. In Fig.9 the fixed wave area is where the change in depth, δ_2, is not negligible and it is equal to the hydraulic jump. Using (1.3) and (1.4) and knowing V_1 and y_1 we can calculate V_2 and y_2 by applying continuity and momentum across the wave. If we assume C and y in Fig.9 as upstream conditions V_1 and y_1 and put $C-\delta V$ and $y+\delta y$ as downstream conditions V_2 and y_2, correct jump solution could be written as equation:

$$V_1^2 = \frac{1}{2} g y_1 \eta (\eta + 1) \quad (4.5)$$

Where$\eta = y_2/y_1$. Solving this quadratic equation for η and introducing the Froude number $Fr = V_1.\sqrt[-2]{gy_1}$, we obtain:

$$\frac{2y_2}{y_1} = -1 + \sqrt{(1 + 8Fr_1^2)} \tag{4.6}$$

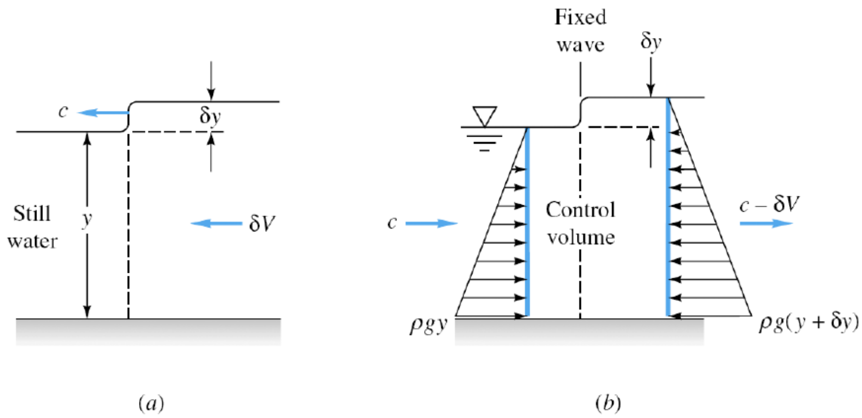

Fig.9, (a) moving wave, non-steady frame; (b) fixed wave, inertial frame of reference [11].

Knowing $V_1y_1 = V_2y_2$ as continuity condition in a wide channel and applying steady flow energy equation, we can calculate the jump dissipation loss:

$$\Delta E = E_1 - E_2 = \left(y_1 + \frac{V_1^2}{2g}\right)\left(y_2 + \frac{V_2^2}{2g}\right) \tag{4.7}$$

Using abovementioned relation for wide channels we have:

$$\Delta E = \frac{(y_2 - y_1)^3}{4y_1y_2} \tag{4.8}$$

Due to second law of thermodynamics it is obvious that if $y_2 > y_1$, the dissipation loss is positive and to have a supercritical upstream flow we should meet $Fr > 1$. On the other hand, to have a subcritical downstream flow it is needed to have $V_2 < V_1$.

4. Conclusion

Using continuum mechanics in different aspects of engineering leadsd to a better understanding of the problems and reveals a better and more thorough solution to those problems. An example could be this study on shallow water equations and hydraulic jump which has utilized the mass and momentum balance laws to solve the problem for shallow water equations and give an exact and applicable final equation. Whith assistance of the results, one can easily predict the flows and currents in the shallow part of ocean and even take care of tsunamis. Besides, using this procedure to predict and analayze the behavior of hydraulic jums leads to a deep understanding of how to deal with this phenamenon in the real time problems.

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Appendix I

Measure of the amount of material in a body which causes wight under gravity is called mass. In continuum mechanics, it is considered that we have a continous distribution of mass in a specific volume of a body and it is the integral of a density field $\rho: \overline{\Omega}_0 \to R^+$, known as mass density. Motion φ cannot affect the body mass M(B) but with a change in body volume in a motion the mass density ρ will change [10]. It can be written:

$$M(B) = \int_{\Omega_t} \rho \, dx$$

Where dx is volume element in $\Omega_t$body configuration. In Fig.10, assume that $\rho\varphi$ and $\rho\psi$ represent mass densities in different paths. As total mass cannot be affected by motion:

$$M(B) = \int_{\varphi(\Omega_0)} \rho\varphi \, dx = \int_{\psi(\Omega_0)} \rho\psi \, dx$$

This is the principle of conservation of mass which means the mass of a body B is constant although the weight of B, gM(B), is variable in different gravity fields where g is constant gravity field.

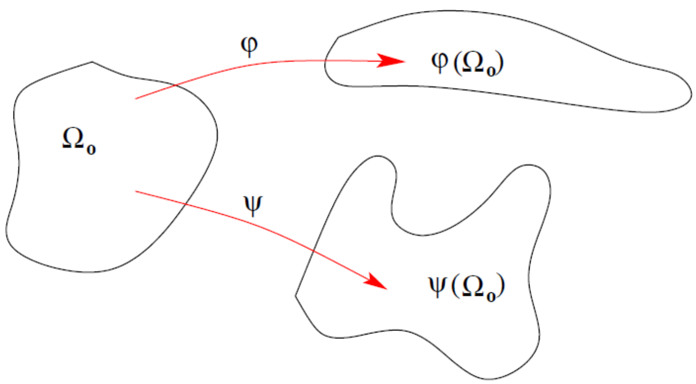

Fig.10, Two motions ψ and φ [13].

For local forms of the orinciple of conservation of mass, consider reference configurational mass density as $\rho_0 = \rho_0(\mathbf{X})$ and mass density in Ω_t configuration as ρ=$\rho(\mathbf{x},t)$. We will have:

$$\int_{\Omega_0} \rho_0(\mathbf{X}) dX = \int_{\Omega_t} \rho(\boldsymbol{x}) dx$$

We know that dx = det $\mathbf{F}(\mathbf{X})$ dX. So we can write:

$$\int_{\Omega_0} \left[\rho_0(\mathbf{X}) - \rho\big(\varphi(\mathbf{X})\big) \det \mathbf{F}(\mathbf{X})\right] dX = 0$$

Equalling the left hand side of the above equation to zero, we will have:

$$\rho_0(\mathbf{X}) = \rho(\boldsymbol{x}) \det \mathbf{F}(\mathbf{X})$$

This is the principle of conservation of mass in the *Lagrangian formulation*. For *Eulerian formulation* we express the invariance of total mass as following:

$$\frac{d}{dt}\int_{\Omega_t} \rho(\boldsymbol{x},t)\,dx = 0$$

By changing the material coordinates:

$$0 = \frac{d}{dt}\int_{\Omega_0} \rho(\boldsymbol{x},t)\det\mathbf{F}(\mathbf{X},t)\,dX = \int_{\Omega_0}\left(\rho\,\dot{\det}\,\mathbf{F} + \dot{\rho}\det\mathbf{F}\right)dX$$

And we know that $(\dot{\bullet}) = d(\bullet)/dt$ and it is good to remember that $\dot{\det}\,\mathbf{F} = \det\mathbf{F}\,\mathrm{div}\,\mathbf{v}$. Using these two remarks, we will have:

$$0 = \int_{\Omega_0} \det\mathbf{F}\left(\rho\,\mathrm{div}\,\mathbf{v} + {}^{\partial\rho}/_{\partial t} + \mathbf{v}.\,\mathrm{grad}\,\rho\right)\,dX$$

And finally we will have:

$${}^{\partial\rho}/_{\partial t} + \mathrm{div}(\rho\mathbf{v}) = 0$$

Appendix II

Material body momentum is a property caused by combination of mass and velocity. About the origin of spatial coordinate system $\boldsymbol{O}$, we can easily derive the *linear momentum* $\boldsymbol{I}(B,t)$ and *angular momentum* $\boldsymbol{H}(B,t)$ by having motion φ, time t, mass density ρ for body mass B as follows [17]:

$$\boldsymbol{I}(B,t) = \int_{\Omega_t} \rho\mathbf{v}\ dx$$

$$\boldsymbol{H}(B,t) = \int_{\Omega_t} \boldsymbol{x}\times\rho\mathbf{v}\,dx$$

Note that the volume element $dx = (dx_1, dx_2, dx_3)$ is in Ω_t configuration. Rates of change for both momenta are very important and in order to calculate these rates we should consider that for any smooth field $\omega = \omega(\boldsymbol{x},t)$:

$$\frac{d}{dt}\int_{\Omega_t}\omega\rho\,dx = \frac{d}{dt}\int_{\Omega_0}\omega(\varphi(\mathbf{X},t),t)\rho(\boldsymbol{x},t)\det\mathbf{F}(\mathbf{X},t)\,dX$$
$$= \int_{\Omega_0}\frac{d\omega}{dt}\rho_0 dX = \int_{\Omega_t}\frac{d\omega}{dt}\rho\,dx$$

So, we will have the linear and angular momentums:

$$\frac{d\boldsymbol{I}(B,t)}{dt} = \int_{\Omega_t}\rho\frac{d\mathbf{v}}{dt}\,dx$$

$$\frac{d\boldsymbol{H}(B,t)}{dt} = \int_{\Omega_t}\boldsymbol{x}\times\rho\frac{d\mathbf{v}}{dt}\,dx$$