On the Generalization of Lami's Theorem

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Abstract - *This present paper gives a broad idea of cyclic nature of the convex polygons having odd number of sides and their application in static analysis of mechanical and structural systems. By using Lami's Theorem we analyze the static equilibrium for three forces system but as the number of forces increase it becomes very difficult in analysis, here in this paper Lami's Theorem is modified and generalized to analyze the system of forces when in static equilibrium and have coplanar, concurrent and non-collinear in nature regardless of their number. In this paper also a sufficient condition is derived for vector/space diagram which helps in determining that whether the vector/space diagram can transform into cyclic convex polygon/n-gon or not and its application in static analysis.*

Key Words: Polygon, Lami's Theorem, Coplanar, Concurrent, Non-Collinear, Static equilibrium, Vector/Space diagram.

1. INTRODUCTION

From the past works we know the magnitudes of three coplanar, concurrent and non-collinear forces, which keeps an object in static equilibrium, with the angles directly opposite to the corresponding forces is given by [1]

$$\frac{a}{sin\alpha} = \frac{b}{sin\beta} = \frac{c}{sin\gamma} \quad \text{... (i)}$$

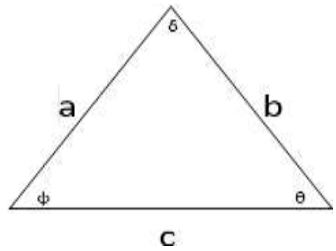
where a, b and c are the magnitude of three coplanar, concurrent and non-collinear forces, which keep the object in static equilibrium, and α, β and γ are the angles directly opposite to the forces A, B and C respectively.

The above theorem is named Lami's theorem after Bernard Lami (1640-1715) [2].

In the Lami's theorem sine rule for triangle's play an essence role which says

$$\frac{a}{sin\theta} = \frac{b}{sin\emptyset} = \frac{c}{sin\delta} = 2R$$

where a, b, and c are the length of the sides of a triangle, and θ, φ, and δ are the opposite angles, while 2R is the diameter of the triangle's circumcircle.

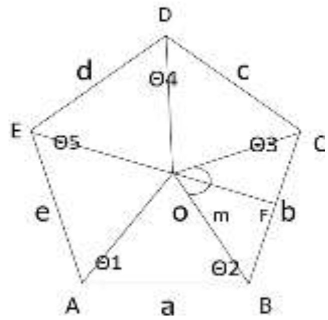


1.1 For 3-gon:

In case of triangles i.e. 3-gon we have sine rule given by equation (i) [3][4]. It is interesting that n-gon (it is requested to reader to treat polygon as convex polygon wherever term polygon is used in this paper) having odd number of sides i.e. for n ϵ odd the sine rule gets the similar form as in case of 3-gon. For time, in this paper brief calculation for 5-gon and 7-gon are discuss in detail and the pattern that the sine rule follows.

1.2 For 5-gon:

Let a 5-gon ABCDE cyclic in nature under circle of radius R and having sides a, b, c, d, e and angles θ_1, θ_2, θ_3, θ_4, and θ_5 as shown in figure. Let a line OF⊥BC is drawn and m = ∠BOC. Then,



Considering the fig. of 5-gon, we have

$$2\pi - \angle EOB = 2\angle EAB = 2\theta_1$$

$$\Rightarrow \quad \angle EOB = 2(\pi - \theta_1) \quad \text{... (ii)}$$

And from ΔFOC,

$$\sin\left(\frac{m}{2}\right) = \frac{b}{2R}$$

Or

$$m = 2\sin^{-1}\left(\frac{b}{2R}\right) \quad \text{... (iii)}$$

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From fig. of 5-gon, we have

$$\angle EOB + \angle m = 2\angle EDC \quad \text{... (iv)}$$

Using equation (ii), (iii) and (iv), we have

$$\theta_1 + \theta_4 = \pi + \sin^{-1}(\frac{b}{2R})$$

Taking sine on both sides, we get

$$\sin(\theta_1 + \theta_4) = -\frac{b}{2R}$$

Similarly,

$$\sin(\theta_1 + \theta_3) = -\frac{e}{2R}$$

$$\sin(\theta_5 + \theta_3) = -\frac{a}{2R}$$

$$\sin(\theta_5 + \theta_2) = -\frac{c}{2R}$$

$$\sin(\theta_4 + \theta_2) = -\frac{d}{2R}$$

From all above equations, we get

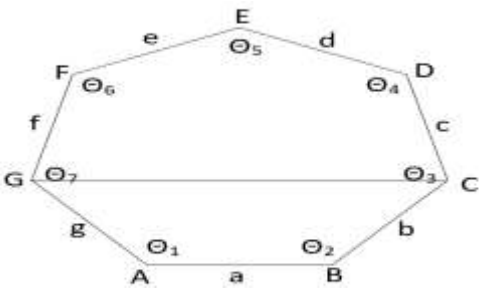
$$\frac{a}{\sin(\theta_3+\theta_5)} = \frac{b}{\sin(\theta_1+\theta_4)} = \frac{c}{\sin(\theta_5+\theta_2)} = \frac{d}{\sin(\theta_4+\theta_2)} = \frac{e}{\sin(\theta_1+\theta_3)} = 2R$$

From above we can conclude for any cyclic 5-gon

$$\frac{corresponding\ sides}{\sin(\sum \theta_{alternate})} = constant$$

1.3 For 7-gon:

Let a 7-gon ABCDEFG, cyclic in nature under a circle of radius R and having sides a, b, c, d, e, f, g and angles θ_1, θ_2, θ_3, θ_4, θ_5, θ_6 and θ_7 as shown in figure. Join GC. Then,



Considering the fig. of 7-gon, we have

$$\theta_5 + \angle FGC = \pi + \sin^{-1}(\frac{c}{2R}) \quad \text{... (v)}$$

And

$$\theta_2 + \angle CGA = \pi \quad \text{... (vi)}$$

Adding equation (v) and (vi)

$$\theta_2 + \theta_5 + \angle FGC + \angle CGA = 2\pi + \sin^{-1}(\frac{c}{2R})$$

Or

$$\theta_2 + \theta_5 + \theta_7 = 2\pi + \sin^{-1}(\frac{c}{2R})$$

Taking both sides sine, we have

$$\sin(\theta_2 + \theta_5 + \theta_7) = \frac{c}{2R}$$

Similarly,

$$\sin(\theta_3 + \theta_5 + \theta_7) = \frac{a}{2R}$$

$$\sin(\theta_4 + \theta_6 + \theta_1) = \frac{b}{2R}$$

$$\sin(\theta_6 + \theta_1 + \theta_3) = \frac{d}{2R}$$

$$\sin(\theta_7 + \theta_2 + \theta_4) = \frac{e}{2R}$$

$$\sin(\theta_1 + \theta_3 + \theta_5) = \frac{f}{2R}$$

$$\sin(\theta_2 + \theta_4 + \theta_6) = \frac{g}{2R}$$

From all above equations, we get

$$\frac{a}{\sin(\theta_3+\theta_5+\theta_7)} = \frac{b}{\sin(\theta_4+\theta_6+\theta_1)} = \frac{c}{\sin(\theta_2+\theta_5+\theta_7)} = \cdots = \frac{g}{\sin(\theta_2+\theta_4+\theta_6)} = 2R$$

From above we can conclude for any cyclic 7-gon,

$$\frac{corresponding\ sides}{\sin(\sum \theta_{alternate})} = constant$$

In the same way for the higher n-gon's we obtain the same pattern i.e. for cyclic polygons when n is odd then

$$\frac{corresponding\ sides}{\sin(\sum \theta_{alternate})} = constant \quad \text{... (vii)}$$

Where $\sum\theta_{alternate}$ denotes the sum of all alternate angles in a n-gon except the angles associated with the corresponding side. For e.g. in case of 5-gon $\sum\theta_{alternate} = \theta_1+\theta_4$ for the side b and θ_2, θ_3 are the angles associated with b therefore didn't take part in $\sum\theta_{alternate}$ for the side b.

2. TRANSFORMATION OF n-gon's INTO VECTOR DIAGRAM AND VICE VERSA

Any n-gon can be transformed into the vector/space diagram such that length of the sides of n-gon represents magnitude of the vectors and the direction of these vectors from each other can be represent as function of interior angles of n-gon's.

Transformation of n-gon into vector diagram can be done by drawing lines (of same magnitude as of sides) parallel to the sides of the n-gon and bringing them at one of the vertices of the n-gon (any vertices can be taken as origin for n-gon for vector diagram transformation, there be only change in the transformation matrix with change of vertices) such that the transformation is optimum and wide to application.

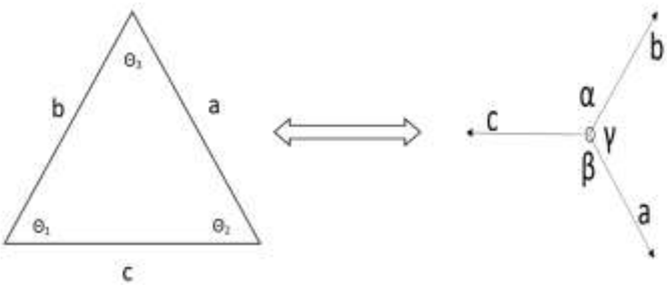
For the time the transformation of 3-gon, 5-gon and 7-gon is presented but it is true and can be manipulated for any n-gon.

2.1 For 3-gon:

Transformation can be done by drawing the lines parallel to the side of the 3-gon and bringing all the sides at one vertices/point. In the below example of ΔABC having sides a, b, c corresponding to angles θ_1, θ_2, θ_3 (or A, B, C) respectively, point B is taken as origin O. These are the steps involved in the transformation,

1. Point B is set as origin O.
2. Line parallel to b is bring at point B such that it makes angle α with c.
3. Line parallel to a is bring at point B such that it makes angles β and γ with c and b respectively as shown in diagram.

The line parallel to a is drawn here considering the optimum transformation and having wide to application.

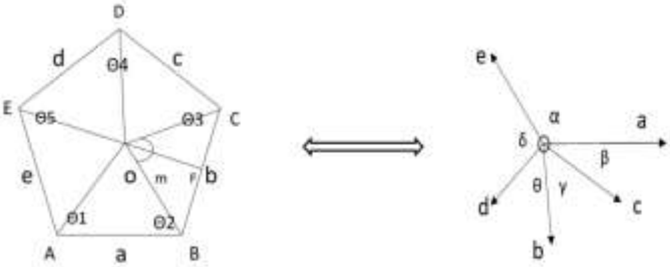


It is easy to obtain from the diagram that the transformation of 3-gon to vector diagram and vice versa can be given by

$$\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\end{pmatrix} = \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\pi-\alpha\\ \pi-\beta\\ \pi-\gamma\end{pmatrix}$$

2.2 For 5-gon:

In case of 5-gon ABCDE, point A is taken as origin and vector transformation of 5-gon is done as shown in diagram.

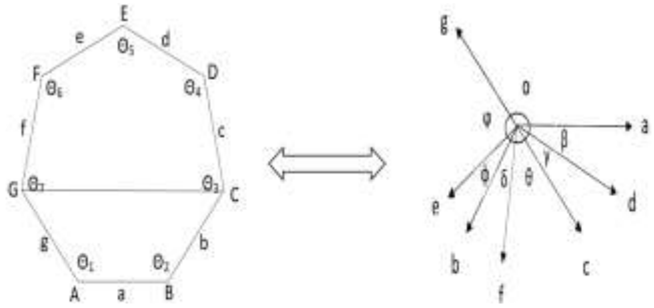


One can obtain in this case that transformation can be given by

$$\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0\\ 0 & 1 & -1 & 0 & 0\\ 0 & 0 & 1 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\alpha\\ \pi+\beta\\ \pi-\gamma\\ \pi+\theta\\ \delta\end{pmatrix}$$

2.3 For 7-gon:

In case of 7-gon ABCDEFG, point A is again taken as origin and vector transformation of 7-gon is done as shown in diagram.



In this case transformation can be given by

$$\begin{pmatrix}\theta_1\\ \theta_2\\ \theta_3\\ \theta_4\\ \theta_5\\ \theta_6\\ \theta_7\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & 0\\ -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & -1/2 & -1/2\\ 1/2 & 1/2 & -1/2 & -1/2 & -1/2 & 1/2 & 1/2\\ 0 & 0 & 1 & 0 & 0 & 0 & 0\\ -1/2 & -1/2 & -1/2 & 1/2 & 1/2 & 1/2 & -1/2\\ 1/2 & 1/2 & -1/2 & 1/2 & -1/2 & -1/2 & 1/2\\ -1/2 & -1/2 & 1/2 & -1/2 & 1/2 & 1/2 & 1/2\end{pmatrix}\begin{pmatrix}\alpha\\ 2\pi+\beta\\ \pi-\gamma\\ 2\pi+\theta\\ 2\pi+\delta\\ 2\pi+\emptyset\\ \pi+\varphi\end{pmatrix}$$

3. CONDITION FOR THE VECTOR/SPACE DIAGRAM TO BE CYCLIC n-gon

Let $S_n(x)$ denotes the function of angles and defined as,

$$S_n(x) = f(\alpha,\beta,\gamma,\theta,\delta,\dots) = \sum_x \theta_{alternate}$$

where α, β, γ, θ, δ, … denotes the angles corresponding to the vector/space diagram of n-gon. Then,

For 3-gon:

$$S_3(a) = \sum\nolimits_a \theta_{alternate} = \pi - \alpha$$
$$S_3(b) = \sum\nolimits_b \theta_{alternate} = \pi - \beta$$
$$S_3(c) = \sum\nolimits_c \theta_{alternate} = \pi - \gamma$$

For 5-gon:

$$S_5(a) = \sum\nolimits_a \theta_{alternate} = \pi + \delta - \gamma$$
$$S_5(b) = \sum\nolimits_b \theta_{alternate} = \alpha + \theta + \gamma$$
$$S_5(c) = \sum\nolimits_c \theta_{alternate} = \beta + \delta + \gamma$$
$$S_5(d) = \sum\nolimits_d \theta_{alternate} = \pi + \alpha - \gamma$$
$$S_5(e) = \sum\nolimits_e \theta_{alternate} = \theta + 2\gamma + \beta$$

In the same way $S_n(x)$ can be obtained for all n-gon.
Equation (vii) in terms of $S_n(x)$ can be given as follow:

$$\frac{x}{\sin(S_n(x))} = constant \qquad \text{… (viii)}$$

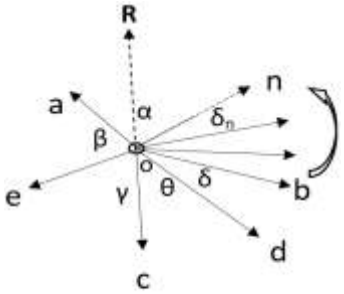
To obtain the condition that the vector diagram is cyclic n-gon or not static analysis relating the co-planar, concurrent and co-linear vectors are being use.

Consider n-gon vector diagram having vectors of magnitude a, b, c, d, e, …, n and transformed angles be α, β, γ, θ, δ, …, δ_n as considered in vector diagram.

Considering the system is in static equilibrium under vector $\vec{a}$, $\vec{e}$, $\vec{d}$, $\vec{b}$, …, $\vec{n}$ and $\vec{c}$. Let $\vec{R}$ be the resultant vector of vectors $\vec{a}$, $\vec{e}$, $\vec{d}$, $\vec{b}$, …, $\vec{n}$ having magnitude **R** and countering the vector $\vec{c}$.

Therefore, if the system is in static equilibrium then magnitude of $\vec{R}$ and $\vec{c}$ must be same i.e.

$$\boldsymbol{R} = c \qquad \text{… (ix)}$$



The magnitude of resultant vector $\vec{R}$ can be given by

$$\boldsymbol{R} = \sum_{x \neq c} x \cos(C_n(x))$$

Where,

$$C_n(x) = g(\alpha, \beta, \gamma, \theta, \delta, \ldots) = \pi - \angle cox$$

From equation (viii) we have,

$$x = k \sin(S_n(x))$$

Where k is some constant.
Putting this value in equation defining **R**, we get

$$\boldsymbol{R} = \sum_{x \neq c} k\, sin(S_n(x)) \cos(C_n(x))$$

from equation (ix), we have

$$\boldsymbol{R} = k \sin(S_n(c))$$

Or

$$\sum_{x \neq c} k\ \sin(S_n(x)) \cos(C_n(x)) = k \sin(S_n(c))$$

Or

$$\sum \sin(S_n(x)) \cos(C_n(x)) = (\cos(C_n(c)) + 1) \sin(S_n(c))$$

Above relation is the required condition for a vector/space diagram to be cyclic n-gon.

4. CONCLUSIONS

In statics, Lamis's Theorem is an equation relating the magnitudes of three co-planar, concurrent and non-collinear forces, which keeps an object in static equilibrium with the angles directly opposite to the corresponding sides.

The Lami's Theorem can be modified for more than three forces without hindering its beauty and in more details. Also the transformation of n-gon to vector diagram acts like the bridge which helps in establishing the relationship.

The statements for the generalized Lami's Theorem can be given as,
"When three or more than three (number of forces be odd) co-planar, concurrent and non-collinear forces acts on an object such that it keeps the object in static equilibrium and the forces follow the cyclic condition then each forces are proportional to the sine of the function of angles between the others forces." According to theorem,

If

$$\sum \sin(S_n(x))\cos\big(C_n(x)\big) = \big(\cos\big(C_n(c)\big)+1\big)\sin(S_n(c))$$

Then for all n belongs odd

$$\frac{x}{\sin(S_n(x))} = constant$$

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