Boundary Layer Flow and Heat Transfer of a Nanofluid over a Wedge

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Abstract - *Two-dimensional boundary layer flow of a nanofluid over a moving wedge in which the model accounts Brownian motion and thermophoresis effects are studied. The governing equations are coupled nonlinear partial differential equations. Using similarity transformations these equations are transformed into nonlinear ordinary differential equations. Thereby the equations are solved numerically by an implicit finite difference scheme-Keller box method. The velocity, temperature and concentration profiles for various physical parameters are represented graphically and are discussed.*

Keywords: nanofluids, boundary layer flows, heat transfer, Keller box method.

1. INTRODUCTION

Boundary layer theory was first introduced by a German Scientist Ludwig Prandtl [1] in his paper *On the motion of a fluid with very small viscosity* presented at the Third International Congress of Mathematicians at Heidelberg on August, 1904. He made a hypothesis that flow of fluid with less viscosity over a solid surface that is smooth can be divided into two regions: near-field which is very thin region close to the surface where the viscous effects are significant *i.e.* the boundary layer region and far-field where the viscosity effects are not significant *i.e.* the potential flow region.

The equations of motion for a viscous fluid were given by Navier [2], Stokes [3] and others. The Navier-Stokes equations define the conservation of momentum. These equations are always solved with the conservation of mass. Solving these equations for a specific boundary conditions gives us the fluid velocity. Later Blasius [4] proposed a similarity solution by considering a flow over a semi-infinite flat plate. Falkner and Skan [5] generalized Blasius solution considering the flow over a wedge. Boundary layer concept that was described by Falkner and Skan include the influence of favorable and adverse pressure gradients. Certain assumptions were considered to develop the mathematical model for the fluid flow in a boundary layer such as the flow is two-dimensional, steady and laminar, the fluid is incompressible and has a large Reynolds number, viscous flow region is thin, at the boundary there is no-slip, no mass flows through the boundary, transport properties of the fluid are uniform, momentum of the fluid and the pressure gradient normal to the boundary are miniscule, static pressure varies only in the direction of the fluid flow in the boundary layer and velocity gradients in normal direction creates large viscous stresses.

Nanofluids means dispersion of nanometer sized metallic particles into the industrial heat transfer fluids(base fluids) and the term nanofluids was first coined by Choi [6]. Earlier when Maxwell's [7] theory of conductivity were available studies were restricted to millimeter or micrometer sized dispersions. But researchers at Argonne National Laboratory found that micrometer sized particles were clogging severely and hence cannot be used for practical purposes. Therefore nanometer sized particles were introduced. Nanofluids are a new class of engineered heat transfer fluids that will enhance the thermal conductivity of the normal fluids without clogging in flow passages.

To study the enhancement of heat transfer by suspending solid particles in fluid, there are two models available in the literature. They are: *single-phase model* wherein both the solid particles and fluid are in thermal equilibrium and the flow will have same velocity and *two-phase model* where the properties of the solid particles as well as the fluid are studied distinctly. There are seven slip mechanisms [8] studied namely inertia, Brownian diffusion, thermophoresis, diffusiophoresis, Magnus effect, fluid drainage and, gravity settling. Out of these seven only Brownian diffusion and thermophoresis slip mechanisms are applicable for laminar flow.

Yacob *et.al.* [9] have studied the flow and heat transfer characteristics over a static or moving wedge with a prescribed surface heat flux in a nanofluid, Mabood *et.al.* [10] theoretically studied the effects of volume fraction of nanoparticles, suction/injection and convective heat and mass transfer parameters for MHD stagnation flow of nanofluids, Sheikholeslami *et.al.* [11] numerically investigated the natural convection heat transfer in a cold outer circular enclosure with a hot inner sinusoidal cylinder that is filled with Cu-nanofluid in the presence of magnetic field, Narahari *et.al.* [12] have discussed the natural convective boundary layer flow of a nanofluid along with the effects of Brownian motion and thermophoresis over an inclined isothermal plate numerically, Chandrashekar *et.al.* [13] used a variational technique: Gyarmati principle to solve the MHD radiative flow over a non-isothermal stretching sheet with Brownian motion and thermophoresis effects and we can find numerous work on boundary layer flow and heat transfer of a nanofluid.

In this paper, we study the effects of Brownian motion and thermophoresis of nanofluid on boundary layer flow. The wedge is moving in the opposite direction to the mainstream in the nanofluid. The governing equations which are nonlinear are solved numerically by Keller box method.

The present paper is structured as follows: § 1 gives the introduction of boundary layer theory and nanofluids. § 2 is dedicated for the mathematical formulation. § 3 offers solution methodology. § 4 presents the results and discussion. § 5 concludes the obtained results.

2. MATHEMATICAL FORMULATION

Consider a steady, two-dimensional laminar boundary layer flow a viscous incompressible nanofluid over a wedge moving with a velocity $U_w(x)$. Here we are considering the Buongiorno model [8] for nanofluids.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad (1)$$

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\nu\frac{\partial^2 u}{\partial y^2} \quad (2)$$

$$u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}+\tau\left(D_B\frac{\partial \varphi}{\partial y}\frac{\partial T}{\partial y}+\frac{D_T}{T_\infty}\left(\frac{\partial T}{\partial y}\right)^2\right) \quad (3)$$

$$u\frac{\partial \varphi}{\partial x}+v\frac{\partial \varphi}{\partial y}=D_B\frac{\partial^2 \varphi}{\partial y^2}+\frac{D_T}{T_\infty}\frac{\partial^2 T}{\partial y^2} \quad (4)$$

subject to the boundary conditions

$$\begin{matrix} at & y=0 & u=U_w x^m, & v=0, & T=T_w, & \varphi=\varphi_w \\ as & y\to\infty & u=U_\infty x^m, & & T=T_\infty, & \varphi=\varphi_\infty \end{matrix} \quad (5)$$

Also we have

$$T=T_w=T_\infty+Cx^r \quad and \quad \varphi=\varphi_w=\varphi_\infty+Dx^s.$$

where u, v are the velocity components in x and y directions respectively, U is the mainstream velocity, T is the temperature of the nanofluid, φ is the volume fraction of the nanoparticle, ν is the kinematic viscosity of the nanofluid, α is the thermal diffusivity of the nanofluid, $\tau=\frac{(\rho C)_p}{(\rho C)_f}$ is the ratio of nanoparticle heat capacity and base fluid heat capacity, D_B is the Brownian diffusion coefficient, D_T is the thermophoresis diffusion coefficient, r, s are the wall temperature and concentration parameters.

Introducing the stream function $(u,v)=\left(\frac{\partial\psi}{\partial y},-\frac{\partial\psi}{\partial x}\right)$, the continuity equation is certainly fulfilled. By making use of the similarity transformations

$$\psi=\sqrt{\frac{2\nu U_\infty x^{m+1}}{m+1}}\,f(\eta), \quad T=(T_w-T_\infty)\theta(\eta)+T_\infty \quad and \quad \varphi=(\varphi_w-\varphi_\infty)\phi(\eta)+\varphi_\infty,$$

$$\eta=\sqrt{\frac{(m+1)U_\infty x^{m-1}}{2\nu}}\,y.$$

in (2) - (5) and simplifying, we get

$$f'''+ff''+\beta(1-f'^2)=0 \quad (6)$$

$$\theta''+Pr(f\theta'+r(\beta-2)f'\theta+N_b\theta'\phi'+N_t\theta'^2)=0 \quad (7)$$

$$\phi'' + Sc(f\phi' + s(\beta - 2)f'\phi) + \frac{N_t}{N_b}\theta'' = 0 \quad (8)$$

with the boundary conditions

$$\begin{aligned} &f(0) = 0, \quad f'(0) = -\lambda, \quad \theta(0) = 1, \quad \phi(0) = 1 \\ &f'(\infty) = 1, \quad \theta(\infty) = 0, \quad \phi(\infty) = 0 \end{aligned} \quad (9)$$

where,

$f = \quad f(\eta), \theta = \theta(\eta), \phi = \phi(\eta)$

$\lambda = \quad -\frac{U_0}{U_\infty}$ *is the ratio of mainstream velocity to boundary velocity,*

$\beta = \quad \frac{2m}{m+1}$ *is the Hartree pressure gradient parameter,*

$Pr = \quad \frac{\nu}{\alpha}$ *is the Prandtl number which is the ratio of momentum diffusivity to thermal diffusivity,*

$Sc = \quad \frac{\nu}{D_B}$ *is the Schmidt number which is the ratio of momentum diffusivity to Brownian diffusivity,*

$N_b = \quad \frac{\tau D_B(\varphi_w - \varphi_\infty)}{\nu}$ *is the Brownian motion parameter and*

$N_t = \quad \frac{\tau D_T(T_w - T_\infty)}{\nu T_\infty}$ *is the thermophoresis parameter.*

3. SOLLUTION METHODOLOGY

The system (6) - (9) that describes the flow of nanofluid in two-dimensional boundary layer and heat transfer is highly nonlinear and coupled which cannot be solved in its present form. However the numerical solution is possible. Keller box method is an implicit finite difference scheme that has second order accuracy and is unconditionally stable. Detailed explanations of the method can be referred in Cebeci and Bradshaw [14] and Keller [15]. The system of differential equations (6) - (8) are converted into first order differential equations as:

$$f' = g$$

$$g' = q$$

$$\theta' = w$$

$$\phi' = z$$

$$q' = \beta g^2 - fq - \beta$$

$$w' = Pr(r(2-\beta)g\,\theta - fw - N_b wz - N_t w^2)$$

$$z' = Sc(s(2-\beta)g\,\phi - fz) - \frac{N_t}{N_b}w' \quad (10)$$

and boundary conditions becomes

$$f(0) = 0, \quad g(0) = -\lambda, \quad \theta(0) = 1, \quad \phi(0) = 1$$

$$g(\infty) = 1, \quad \theta(\infty) = 0., \quad \phi(\infty) = 0 \quad (11)$$

We use backward finite-difference to the system (10) - (11), we get

$$\begin{aligned}
f_j - f_{j-1} &= \frac{h}{2}(g_j + g_{j-1}) \\
g_j - g_{j-1} &= \frac{h}{2}(q_j + q_{j-1}) \\
\theta_j - \theta_{j-1} &= \frac{h}{2}(w_j + w_{j-1}) \\
\phi_j - \phi_{j-1} &= \frac{h}{2}(z_j + z_{j-1}) \\
q_j - q_{j-1} &= \frac{h}{2}\left(\frac{\beta}{2}(g_j + g_{j-1})^2 - \frac{1}{2}(f_j + f_{j-1})(q_j + q_{j-1}) - 2\beta\right) \\
w_j - w_{j-1} &= \frac{h}{2}Pr\left(\frac{r(2-\beta)}{2}(g_j + g_{j-1})(\theta_j + \theta_{j-1}) - \frac{1}{2}(f_j + f_{j-1})(w_j + w_{j-1})\right. \\
&\left. -\frac{N_b}{2}(w_j + w_{j-1})(z_j + z_{j-1}) - \frac{N_t}{2}(w_j + w_{j-1})^2\right) \\
z_j - z_{j-1} &= \frac{h}{2}Sc\left(\frac{s(2-\beta)}{2}(g_j + g_{j-1})(\phi_j + \phi_{j-1}) - \frac{1}{2}(f_j + f_{j-1})(z_j + z_{j-1})\right) - \frac{N_t}{N_b}(w_j - w_{j-1})
\end{aligned} \qquad (12)$$

The above system of equations (12) are nonlinear algebraic equations. To linearize them we make use of Newton's linearization technique that is defined as

$$a_j^{(k+1)} = a_j^{(k)} + \Delta a_j^{(k)} \quad (13)$$

where $a = [f \quad g \quad q \quad \theta \quad w \quad \phi \quad z]^{\intercal}$. On substituting (13) into the system (12), we obtain the linear system of algebraic equations which can be conveniently written in the matrix form as

$$AD = R \quad (14)$$

where A is a block tridiagonal matrix and their elements are again a matrix of order 7,

$$D = \left[\begin{matrix}\Delta q_0 & \Delta w_0 & \Delta z_0 & \Delta f_1 & \Delta q_1 & \Delta w_1 & \Delta z_1 \dots \Delta g_{N-1} & \Delta\theta_{N-1} & \Delta w_{N-1} & \Delta f_N \\ \Delta q_N & \Delta w_N & \Delta z_N\end{matrix}\right]^{\intercal},$$

$R = [R_1 \quad R_2 \dots R_N]^{\intercal}$. The tridiagonal matrix is given by

$$\begin{bmatrix}
[A_1] & [C_1] & 0 & \dots & \dots \\
 & [A_2] & [C_2] & 0 & \dots \\
0 & \dots & \dots & \dots & \dots \\
0 & \dots & [B_{N-1}] & [A_{N-1}] & [C_{N-1}] \\
0 & \dots & \dots & [B_N] & [A_N]
\end{bmatrix}$$

where

$$A_1 = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
-d & 0 & 0 & 0 & -d & 0 & 0 \\
0 & -d & 0 & 0 & 0 & -d & 0 \\
0 & 0 & -d & 0 & 0 & 0 & -d \\
c_{12} & 0 & 0 & c_{13} & c_{11} & 0 & 0 \\
0 & c_{21} & c_{29} & c_{26} & 0 & c_{20} & c_{28} \\
0 & c_{33} & c_{31} & c_{34} & 0 & c_{32} & c_{30}
\end{bmatrix}, A_j = \begin{bmatrix}
-d & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
c_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\
c_{24} & c_{22} & 0 & 0 & 0 & 0 & 0 \\
c_{38} & 0 & c_{36} & 0 & 0 & 0 & 0
\end{bmatrix},$$

$$B_j = \begin{bmatrix}
0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -d & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -d & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -d \\
0 & 0 & 0 & c_{14} & c_{12} & 0 & 0 \\
0 & 0 & 0 & c_{27} & 0 & c_{21} & c_{29} \\
0 & 0 & 0 & c_{35} & 0 & c_{33} & c_{31}
\end{bmatrix}, C_{j-1} = \begin{bmatrix}
-d & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 \\
c_{15} & 0 & 0 & 0 & 0 & 0 & 0 \\
c_{24} & c_{22} & 0 & 0 & 0 & 0 & 0 \\
c_{38} & 0 & c_{36} & 0 & 0 & 0 & 0
\end{bmatrix}$$

where $j = 2,3,4, \ldots$. The tridiagonal matrix can be solved by using the factorization method. The solutions of D which is obtained from (14) needs to be updated by using (13) at each iteration process till it is convergent.The error tolerance is set to 10^{-6}.

4. RESULTS and DISCUSSION

Boundary layer flow and heat transfer over a moving wedge of a nanofluid is considered using the Buongiorno model wherein both solid nanoparticles and fluid medium will be dealt separately *i.e.* the governing equations will have two equations on conservation of mass: one for the base fluid and the other for solid nanoparticles, one equation on conservation of momentum and one equation on conservation of energy. The governing equations are nonlinear and coupled, we are solving them by numerical technique which is implicit finite difference scheme: Keller box method. The results obtained numerically are shown graphically.

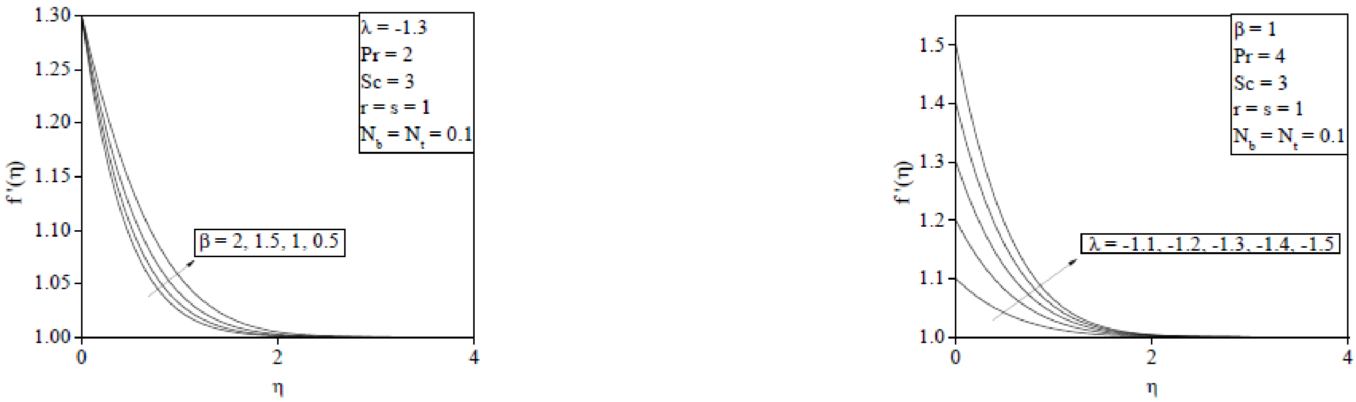
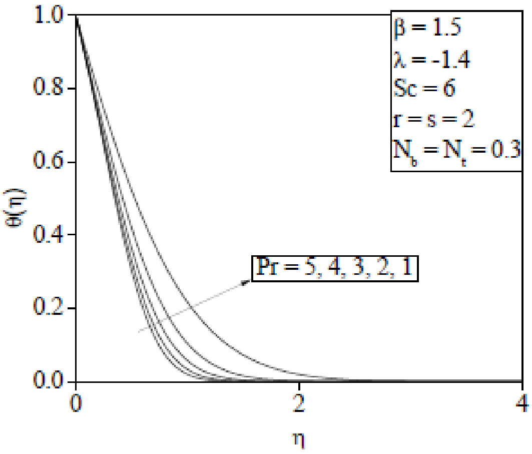


Figure 1: Variation of velocity profiles with η for different β and λ respectively.



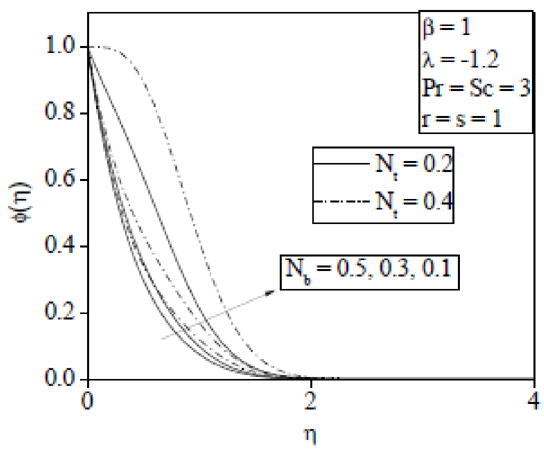


Figure 2: Temperature profiles Figure 3: Concentration profiles

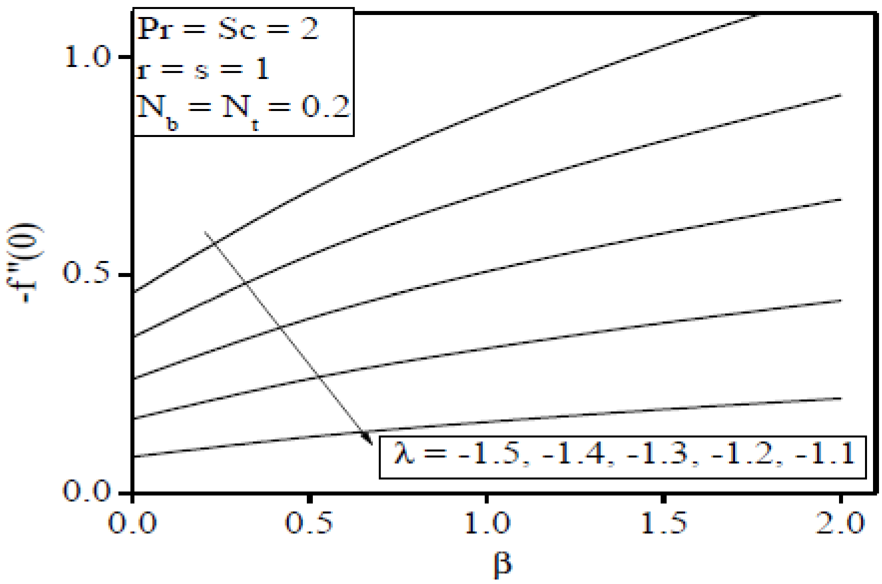


Figure 4: Variation of skin friction with β.

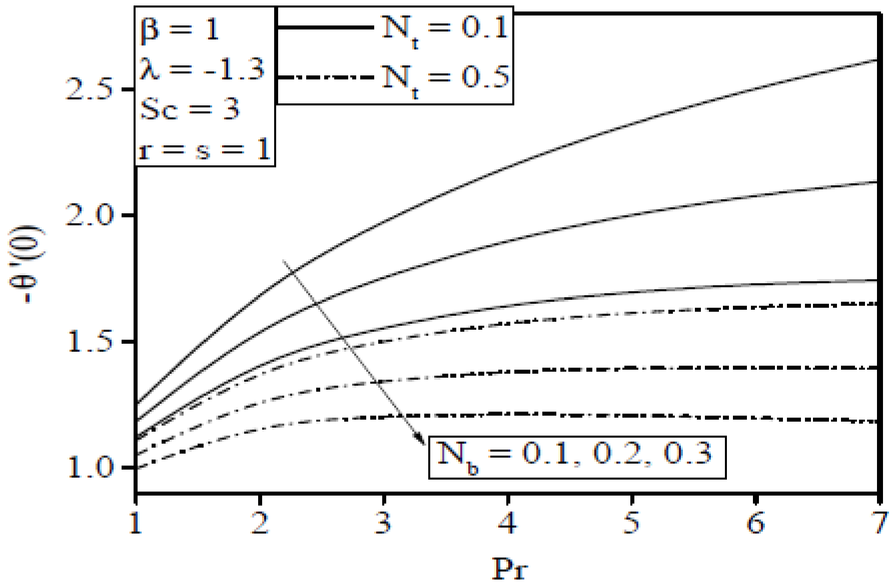


Figure 5: Variation of temperature gradient with Pr.

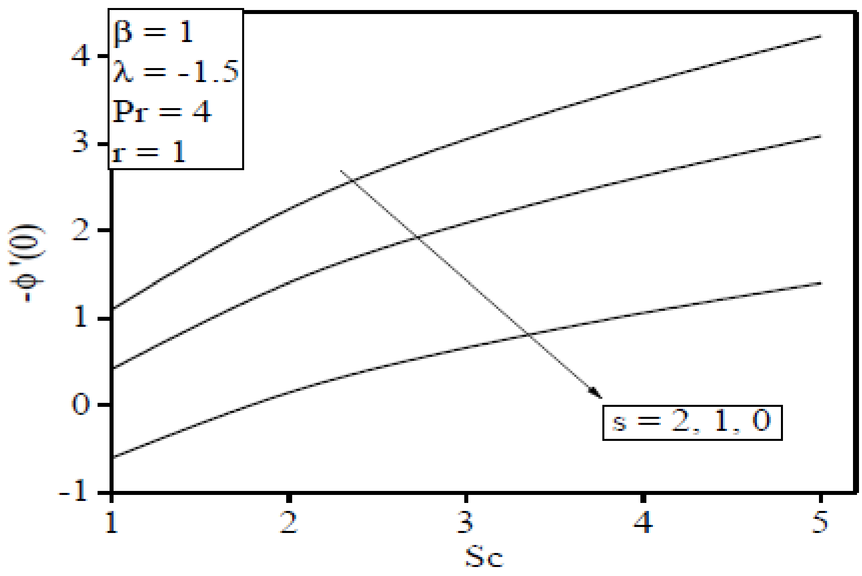


Figure 6: Variation of concentration gradient with Sc.

In figure 1 we observe variation of velocity profiles for different β and λ respectively. 0 describes as the pressure gradient parameter β increases the flow gets accelerated therefore Reynolds number increases and velocity of the nanofluid particles increases and hence boundary layer thickness increases. Figure 0 explains for increasing stretching parameter λ, the wedge velocity is quite higher than the mainstream velocity therby increasing the velocity of the nanofluid particles. For increasing Prandtl number in figure 2, the thermal boundary layer thickness is found to be thinner. Figure 3 describes the variation of concentration $\phi(\eta)$ with η for different thermophoresis (N_t) and Brownian diffusion (N_b) parameters. As N_b decreases, the concentration boundary layer decreases. Same trend is observed for N_t. As the pressure gradient parameter β increases in figure 4, we notice that skin-friction increases . Figure 5 describes that for small Prandtl number there is a substantial increase in the thermal gradient until 3 and N_t kept smaller. However for increasing Prandtl number and higher value of N_t there is no much variation in the thermal gradient. For smaller values of N_t we anticipate the Prandtl number tending to infinity there is a gradual variation in the thermal gradient. From figure 6 we observe that for smaller values of Schmidt number there is an increases till 5 in the concentration gradient and when Schmidt number tends to infinity gradually the profiles are almost flat which is not shown in the figure.

5. CONCLUSIONS

In this paper, we have studied the two-dimensional boundary layer flow and heat transfer over a wedge using Buongiorno model for a nanofluid. The governing equations are solved numerically using Keller-box method. The obtained results are depicted graphically in terms of velocity, temperature, concentration, skin-friction, temperature gradient and concentration gradient. We see that as *Pr* decreases the thermal boundary layer increases and as *Sc* increases the concentration boundary layer becomes thinner.

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