2-step perfect domination on graphs

S.Sahaya Jasmine[1], S.M.Ambika[2]

[1]*M.Phil Scholar, Department of Mathematics, S.T.Hindu college,Nagercoil,Tamil nadu,India*

[2]*Research Scholar, Department of Mathematics, S.T.Hindu college,Nagercoil,Tamil nadu,India*

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Abstract: *In this paper, we introduce another new domination parameter called, the 2-step perfect domination of a graph with real life applications. Let G be a graph. A subset S of vertices in a graph G is a 2-step perfect dominating set if every vertex $v \in V - S$, there exists one vertex $u \in S$ such that $d(u,v) = 2$ and $|N(v) \cap S| = 1$. The 2-step perfect dominating number is the minimum cardinality of a 2-step perfect dominating set of G. The 2-step perfect dominating set of G is found for path, cycle, ladder graph, helm graph, closed helm graph, gear graph, web graph. Also we introduce, subdivision of 2-step perfect dominating set.*

Keywords : **Domination number, Perfect domination set, 2-step domination, 2-step perfect domination set, Subdivision.**

1.INTRODUCTION

Throughout this paper, we consider only finite simple undirected connected graph. . Let $G = (V, E)$ be a graph with vertex set V and edge set E, such that $|V| = n$ $|E| = m$. Degree of a vertex G is denoted by $d(v)$,the maximum degree of a graph G is denoted by $\Delta(G)$. A path on n vertices is denoted by P_n. A cycle of length n is denoted by C_n.

The ladder graph L_n is a planar undirected graph with 2n vertices and 3n-2 edges. The ladder graph can be obtained as the Cartesian product of two path graphs, one of which has only one edge. The helm H_n is a graph obtained from wheel W_n by attaching a pendant edge to each of its rim vertices. The closed helm CH_n is the graph obtained from a helm by joining each pendant vertex to form a cycle. The gear graph is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle.

The corona $G_1 \odot G_2$ of two graphs G_1(with n_1 vertices and m_1 edges) and G_2 (with n_2 vertices and m_2 edges) is defined as the graph obtained by talking one copy of G_1 and n_1 copies of G_2, and then joining the i^{th} vertex of G_1 with an edge to every vertex in the i^{th} copy of G_2. A subdivision of an edge $e = uv$ of a graph G is the replacement of the edge e by a path (u, v, w). The graph obtained from G by subdividing every edge e of G exactly once is called the subdivision graph of G and is denoted by $S(G)$.

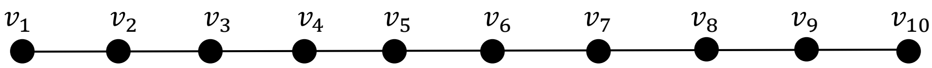
In our literature survey, we are able to find many authors have introduced various new parameters by imposing conditions on the dominating sets. In that sequence, the concept of connectedness plays an important role in any network.

A dominating set S is a subset of V such that every vertex in $V-S$ is adjacent to at least one vertex in S. Further, S is a perfect dominating set if every vertex of $V-S$ is adjacent to exactly one vertex in S. The perfect domination number denoted $\gamma_p(G)$ is the minimum cardinality of a perfect dominating set in G. The perfect dominating set was introduced by by Cockayne et.al. A subset $S \subseteq V$ is a 2-step dominating set if for every vertex $v \in V-S$, there exists $u \in S$ such that distance between u and v is 2. The minimum cardinality of a 2-step dominating set of G is a 2-step dominating number of G, it is denoted by $\gamma_{2s}(G)$. 2-step domination was introduced by Chartrand et. al.

Now we introduce a new domination parameter called 2-step perfect domination. A dominating set is a 2-step perfect dominating set, if for every vertex $v \in V-S$, there exists one vertex $u \in S$ such that $d(u,v)=2$ and $|N(v) \cap S| = 1$. The minimum cardinality of 2-step perfect dominating set is denoted by $\gamma_{2sp}(G)$. 2-step domination was introduced by Chartrand et. al.

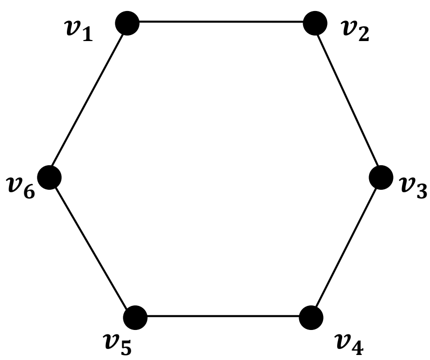
2.EXAMPLES

2.1 Consider the path P_{10},



$S = \{v_1, v_4, v_7, v_{10}\}$ is the 2-step perfect domination set of P $_{10,}\gamma_{2sp}(G) = 4$

2.2 Consider the cycle $C_{6,}$



$S = \{v_3, v_6\}$ is the 2-step perfect domination set of $C_{6,\,,}\gamma_{2sp}(G) = 2$.

3.MAIN RESULTS

Definition 3.1

For a graph $G = (V, E)$, a set $S \subseteq V$ is a ***perfect dominating set*** if for every vertex $v \in V - S$, $|N(v) \cap S| = 1$.

Definition 3.2

A set S of vertices G is called a ***2-step domination set*** if for every vertex $v \in V - S$, there exists atleast one vertex $u \in S$ such that $d(u, v) = 2$.

Definition 3.3

A dominating set S is 2-step perfect dominating set if for every vertex $v \in V - S$, there exists one vertex $u \in S$ such that $d(u, v) = 2$ and $|N(v) \cap S|$=1. The minimum cardinality of ***2-step perfect domination*** is denoted by $\gamma_{2sp}(G)$.

Observation 3.4

For the path P_n $(n \geq 4)$,

$$\gamma_{2sp}(p_n) = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & if\ n\ is\ odd, n \notin 5m \\ \frac{n}{2} & if\ n\ is\ even \\ \left\lfloor \frac{n}{2} \right\rfloor - 1 & if\ n \in 5m,\ m = 1,3,5, \ldots \end{cases}$$

Remark 3.5

$\gamma_{2p}(G) \leq \gamma_{2sp}(G)$

For, Let S be a minimum 2-step perfect dominating set of G.

Let v $\in V - S$. Then $|N(v) \cap S| = 1$

That is every vertex $v \in V - S$,there exists at least one vertex in u$\in S$ such that d(u,v)=2.

Therefore S is a 2-step dominating set of G.

Therefore $\gamma_{2p}(G) \leq \gamma_{2sp}(G)$

Theorem 3.6

For any cycle C_m $(m \geq 4)$,

$$\gamma_{2p}(C_m) = \begin{cases} \frac{m}{3} & m = 3n, n \geq 2 \\ \left\lceil \frac{m}{3} \right\rceil & m = 3n+1,\ n \in N \\ \left\lceil \frac{m}{2} \right\rceil & m = 3n+2, n \geq 2 \end{cases}$$

Proof:

Case (i): Let $G = C_{3n}, n \geq 2$

Let $\{v_1, v_2, v_3, \ldots \ldots . v_{3n}\}$ be the vertices of $V(G)$. $S = \{v_1, v_4, v_7, \ldots . v_{3n-2}\}$ is a dominating set and $V - S = \{v_2, v_3, v_5, v_6, \ldots \ldots v_{3n-1}, v_{3n}\}$. Every vertex in $V - S$ is exactly one vertex in S. Hence S is perfect dominating set and also every vertex in $V - S$ has atleast one vertex in S at 2-step distance. Hence S is the 2-step perfect dominating set. Therefore $\gamma_{2sp}(C_{3n}) = \frac{m}{3}$.

Case (ii) : Let $G = C_{3n+1}, n \in N$

Let $V(G) = \{v_1, v_2, v_3, \ldots \ldots v_{3n}, v_{3n+1}\}$

Now $= \{v_1, v_4, v_7, \ldots \ldots . v_{3n+1}\}$. Here S is a perfect dominating set and also S is the 2-step dominating set of C_{3n+1}. Therefore the 2-step perfect dominating set S of C_{3n+1}, $\gamma_{2sp}(C_{3n+1}) = \left\lceil \frac{m}{3} \right\rceil$

Case (iii) : Let $G = C_{3n+2}, n \geq 2$

Let $V(G) = \{v_1, v_2, \ldots \ldots . v_{3n}, v_{3n+1}, v_{3n+2}\}$

Now $S = \{v_1, v_4, v_7 \ldots \ldots . v_{3n+1}, v_{3n+2}\}$ is a 2-step perfect dominating set of G. That is $|s| = \left\lceil \frac{m}{2} \right\rceil$

Since $\left\lceil \frac{m}{2} \right\rceil = \gamma_{2s}(C_{3n+2}) = \gamma_{2sp}(C_{3n+2})$

Then $\gamma_{2s}(C_{3n+2}) = \gamma_{2sp}(C_{3n+2})$

Observation 3.7

For any cycle C_m ,($m = 5$), $\gamma_{2sp}(C_m) = \left\lceil \frac{m}{2} \right\rceil$

Let $\{v_1, v_2, v_3, v_4, v_5\}$ be the vertices of C_5 . $S = \{v_1, v_2, v_5\}$ is the dominating set of G and $V - S = \{v_3, v_4\}$. v_3 and v_4 intersects with exactly one vertex in S. v_3 and v_4 has 2-step distance with at least one vertex in S. Hence S is a 2-step perfect dominating set. In particular, $\gamma_{2sp}(C_m) = \left\lceil \frac{m}{2} \right\rceil$.

Definition 3.8

The ***helm H_n*** is a graph obtained from wheel W_n by attaching a pendant edge to each of its rim vertices.

Theorem 3.9

For the helm H_n $(n \geq 3)$, $\gamma_{2sp}(H_n) = n + 1$

Proof:

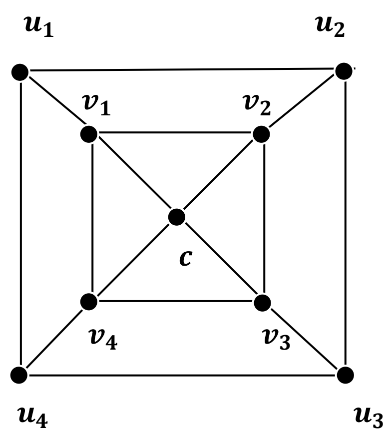
The helm H_n has $2n + 1$ vertices. The apex vertex and n rim vertices form the minimum dominating set of helm H_n . Every vertex in $V - S$ intersect with exactly one vertex in the dominating set S. Hence S is the perfect dominating set and every vertex in $V - S$ has the apex vertex at 2-step distance. Hence S is a 2-step perfect dominating set of H_n. Threfore , $\gamma_{2sp}(H_n) = n + 1$.

Definition 3.10

The ***closed helm CH_n*** is the graph obtained from a helm by joining each pendant vertex to form a cycle.

Illustration 3.11

The closed helm CH_4 is shown in figure



Theorem 3.12

For the closed helm CH_n (n$\geq$ 3), $\gamma_{2sp}(CH_n) = n + 1$.

Proof :

Label the vertices of the closed helm graph CH_n as follows. Denote the vertices of the innermost cycle of H_n successively $v_1, v_2, v_3 \ldots v_n$. Then denote the apex vertex of CH_n as c. Now , to attain minimum cardinality , every 2-step perfect dominating set of CH_n should contain the vertex c because

$\deg(c) = n = \Delta(\mathrm{CH_n})$

The vertex c and the inner most cycle vertices dominate themselves as well as all vertices of outer most cycle of CH_n . Every $v \in V - S$, there exist atleast one vertex $u \in S$ such that

$d(u, v) = 2$ and $|N(v) \cap S| = 1$.

Therefore, for any 2-step perfect dominating set S of H_n, $|S| \geq n$ implying that $\gamma_{2sp}(CH_n) = n + 1$.

Theorem 3.13

For the corona graph $C_m \circ P_n$ $(n \geq 4)$, $\gamma_{2sp}(C_m \circ P_n) = m$.

Proof:

Let $v(C_m) = \{v_1, v_2, v_3, \ldots v_m\}$. Let P_{mn} denote the m^{th} copy corresponding to the path P_n.

$V(C_m \circ P_n) = \{v_1, v_{11}, v_{12}, \ldots v_{1n}, v_2, v_{21}, \ldots v_{2n} \ldots v_m, v_{m1}, \ldots v_{mn}\}$.

We prove this result by induction on m.

Suppose $m = 4$, Then $S = \{v_1\}$ dominate every vertices on the cycle C and its corresponding paths. Therefore S is the minimal dominating set of $C_m \circ P_n$.

For every vertex v belong to $V - S$, $|N(v) \cap S| = 1$.Then S is a perfect dominating set. Also every vertex $v \in V - S$, there exists at least one vertex $u \in S$ such that $d(u, v) = 2$. Thus S is a 2-step dominating set .

Therefore S is a γ_{2sp} set.

Let us assume this result true for $m - 1$.

i.e, $\gamma_{2sp}(C_{m-1} \circ P_n) = m - 1, n \geq 2$

Let us prove for m.

Let $\{v_1, v_2, v_3, \ldots v_m\}$ be the vertices of C_m .Since the result is true for $m - 1$ and for m we need to choose at least one more vertex so that the vertices of m^{th} copy of the path P_n can be dominated by the m^{th} vertex . $S = \{v_1, v_2, v_3, \ldots, v_m\}$ is the minimal dominating set and the vertices $v \in V - S$ has $|N(v) \cap S| = 1$. Every vertex in $V - S$ has 2-step distance with S.

Then $d(u, v) = 2$.

Therefore S is a γ_{2sp} set.

Hence $\gamma_{2sp}(C_m \circ P_n) = m$

This result is true for all m.

Theorem 3.14

$$\text{For } L_n\ (n \geq 2),\ \gamma_{2sp}(L_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & if\ n \geq 3,\ \ n\ is\ odd \\ \frac{n}{2} + 1 & if\ n \geq 2,\ \ n\ is\ even \end{cases}$$

Proof:

Case (i) : n is odd

Let $\{v_1., v_6, v_9 \ldots\ldots v_{n-1}\}$ be the dominating set L_n and $V - S = \{v_2, v_3, \ldots, v_{n-3}, v_{n-2}\}$. Every vertex in $V - S$ is dominated by exactly one vertex in S and it has 2-step distance between S and $V - S$. So S becomes the 2-step perfect dominating set.

In general $n = 2k + 1$ where $= 1,2,3 \ldots, n$.

Therefore , $\gamma_{2sp}(L_n) = \left\lceil \frac{n}{2} \right\rceil$.

Case (ii): n is even

Let $S = \{v_1, v_6, v_9, \ldots, v_n\}$ is a dominating set. Every vertex in V-S satisfy the 2-step dominating set which is perfect . In general $n = 2k, k = 1,2,3, \ldots., n$

Hence , $\gamma_{2sp}(L_n) = \frac{n}{2} + 1$.

Definition 3.15

The ***gear graph*** is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle.

Theorem 3.16

For the gear graph G_n (n≥ 4) , $\gamma_{2sp}(G_n) = \left\lceil \frac{n}{2} \right\rceil + 1$

Proof:

Let c denote the apex vertex of wheel W_n .The gear graph G_n has $2n - 1$ vertices and $3(n - 1)$ edges. The gear graph G_n contains the outer cycle $2n - 1$ and

$$V(G_n) = V(C_{2(n-1)}) \cup \{c\}$$

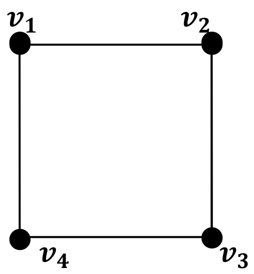
Therefore atleast $\left\lceil \frac{n}{2} \right\rceil + 1$ vertices are essential to dominate all vertices of G_n . Moreover , these vertices also dominate the vertex c .Since every vertex $v \in V - S$, there exists at least one vertex $u \in S$ such that $d(u, v) = 2$ and $|N(v - S) \cap S| = 1$. Hence for any 2-step perfect dominating set S of G_n , $\gamma_{2sp}(G_n) = \left\lceil \frac{n}{2} \right\rceil + 1$.

Definition 3.17

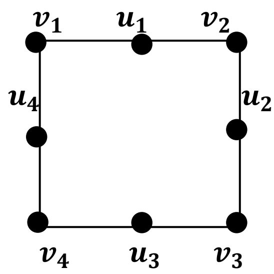
A subdivision of an edge $e = uv$ of a graph G is the replacement of the edge e by a path (u, v, w). The graph obtained from G by subdividing every edge e of G exactly once is called ***the subdivision graph*** of G and is denoted by $S(G)$

Example 3.18

Consider the cycle c_4,



$\boldsymbol{G}$



$\boldsymbol{S(G)}$

Theorem 3.19

For any positive integer $n \geq 2, \gamma_{2sp}\left(S(K_{1,n})\right) = n$+1

Proof :

Let u be the central vertex of $K_{1,n}$. Let $\{v_1, v_2, v_3, \dots, v_n\}$ be the vertices adjacent with u. Each edge is subdivided into two edges uu_i and $u_iv_i, i = 1,2,3, \dots., n$ as shown in figure . Then $S(k_{1,n})$ will have $2n + 1$ vertices and $2n$ edges. Every newly added vertex u_i and u dominates $v_i(1 \leq i \leq n)$. Hence the set $S = \{u, u_1, u_2, u_3, \dots, u_n\}$ is a dominating set of $S(k_{1,n})$. If we delete a vertex u_i from S then $S - \{u_i\}$ will not dominate $S(k_{1,n})$. If we delete a vertex u_i from S then $S - \{u_i\}$ will not dominate $S(k_{1,n})$ and so S is a minimum dominating set of $S(k_{1,n})$. For every $v \in V - S$, there exists atleast one vertex $u \in S$ such that $d(u, v) = 2$ and $|N(v) \cap S| = 1$. Hence every vertex of S is a 2-step perfect dominating set of $S(k_{1,n})$. Therefore $\gamma_{2sp}\left(S(K_{1,n})\right) = n + 1$.

4.CONCLUSION

In this paper, we computed the exact values of strong blast domination for some graphs, subdivision graph, corona product of some graphs. Also we derive several general results on this domination parameter.

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