Certain Differential Coefficients and Convolution Sums Involving Series Identities

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***Abstract*—**Ramanujan recorded different levels of beautiful Eisenstein series relations in his notebooks. Further, Shaun cooper in his book noted several identities involving Eisenstein series and theta functions. In this paper, certain differential identities have been established, which involves eta-functions, Eisenstein series of different levels and the series identities. Further, the convolution sum has been evaluated using Eisenstein series of level 5 and 7.

Keywords- Eisenstein series, Dedekind eta-function, Convolution sum 2010 Mathematics Subject Classification: 11M36, 11F20

1. Introduction

Differential equations and Convolution sums play an important role in computational mathematics. Ramanujan[1] in his book recorded certain differential equations involving theta functions. Berndt B. C. [2], in his paper revealed the importance of constructing differential equations involving eta-functions and Eisenstein series. They have formed certain differential equations to prove the identities of order 14 and 35 in Section 8, 9 and 10. Recently, Vidya H. C. and Srivatsa Kumar B. R. [3] established certain differential equations involving Eisenstein series and theta function identities. Also, they have evaluated discrete Convolution sums using Eisenstein series of different levels recorded by Shaun Cooper.

The present study provides an adequate method to construct differential equations involving eta-functions and series identities, which are achieved by adopting some of the Eisenstein series relations recorded by Cooper S. [4]. In Section 3, we have formed certain differential equations and in Section 4, we have evaluated convolution sums by using Eisenstein series of level 5 and 7 and the identity of Glaisher [5]. Section 2 is dedicated to record some preliminary results.

2. Preliminaries

Definition 2.1.*[1]* For any complex a and q with $|q| < 1$, the q-series is defined by

$$(a;q)_\infty := \prod_{x=0}^{\infty}(1-aq^x).$$

For $|ab| < 1$, Ramanujan's general theta-function [1, p.35] is given by

$$f(a,b) := \sum_{x=-\infty}^{\infty} a^{x(x+1)/2}\, b^{x(x-1)/2} = (-a,-b,ab;ab)_\infty.$$

The special case of a theta function recorded by Ramanujan is defined by

$$f(-q) := f(-q,-q^2) = \sum_{x=-\infty}^{\infty}(-1)^x q^{x(3x-1)/2} = (q;q)_\infty = q^{-1/24}\eta(q). \qquad (1)$$

Definition 2.2. *[1]* The Ramanujan-type Eisenstein series are defined by

$$P(q) := 1-24\sum_{x=1}^{\infty}\frac{xq^x}{1-q^x} = 1-24\sum_{x=1}^{\infty}\sigma_1(x)q^x,$$

$$Q(q) := 1+240\sum_{x=1}^{\infty}\frac{x^3q^x}{1-q^x} = 1+240\sum_{x=1}^{\infty}\sigma_3(x)q^x.$$

Note that, we denote $P(q^n) = P_n$.

Definition 2.3. *[4]* The following power series identity

$$z = \sum_{n=o}^{\infty} h(n) X^n$$

converges in some neighborhood of $X=0$.

Lemma 2.4. *[4]* Let P(q) and $\eta(q)$ be as defined in Definition 1.2 and (1). Then the following identities hold:

Level	$z = \sum_{n=o}^{\infty} h(n) X^n$	X
5	$\frac{5P_5 - P_1}{4} = \sum_{n=0}^{\infty}\left[\binom{2n}{n}\binom{n}{k}^2\binom{n+k}{n}\right]X^n$	$\frac{\eta_1^4\eta_5^4}{z^2}$
6	$\frac{6P_6 - 3P_3 - 2P_2 + P_1}{2} = \sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_k(-8)^{n-k}\binom{n}{k}\binom{k}{l}^3\right]X^n$	$\frac{\eta_1^2\eta_2^2\eta_3^2\eta_6^2}{z^2}$
6	$\frac{6P_6 - 3P_3 + 2P_2 - P_1}{4} = \sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_k\binom{n}{k}^2\binom{2k}{k}\right]X^n$	$\frac{\eta_1^2\eta_2^2\eta_3^2\eta_6^2}{z^2}$
6	$\frac{6P_6 + 3P_3 - 2P_2 - P_1}{6} = \sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_k\binom{n}{k}^3\right]X^n$	$\frac{\eta_1^2\eta_2^2\eta_3^2\eta_6^2}{z^2}$
7	$\frac{7P_7 - P_1}{6} = \sum_{n=0}^{\infty}\left[\sum_k\binom{n}{k}^2\binom{2k}{n}\binom{n+k}{n}\right]X^n$	$\left(\frac{\eta_1^2\eta_7^2}{z}\right)^{3/2}$
8	$\frac{8P_8 - 4P_4 - 2P_2 + P_1}{3} = \sum_{n=0}^{\infty}\left[(-1)^n\binom{2n}{n}\sum_k\binom{n}{k}\binom{2k}{k}\binom{2n-k}{n-k}\right]X^n$	$\frac{\eta_2^4\eta_4^4}{z^2}$
9	$\frac{9P_9 - 6P_3 + P_1}{4} = \sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_k(-3)^{n-3k}\binom{n}{k}\binom{n-k}{k}\binom{n-2k}{k}\right]X^n$	$\frac{\eta_3^8}{z^2}$
10	$\frac{10P_{10} + 5P_5 - 2P_2 - P_1}{12} = \sum_{n=0}^{\infty}\left[\sum_k\binom{n}{k}^4\right]X^n$	$\left(\frac{\eta_1\eta_2\eta_5\eta_{10}}{z}\right)^{4/3}$

3. CONSTRUCTION OF DIFFERENTIAL EQUATIONS

Theorem 3.1. If

$$v := \frac{1}{q^{1/12}} \frac{f_2 f_3}{f_1 f_6}$$

then the following differential identity holds:

$$\frac{q}{v}\frac{dv}{dq} + \frac{1}{12}\sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_{k}(-8)^{n-k}\binom{n}{k}\binom{k}{l}^3\right]X^n = 0,$$

where $X = \dfrac{\eta_1^2\eta_2^2\eta_3^2\eta_6^2}{z^2}$ and $z = \dfrac{6P_6 - 3P_3 - 2P_2 + P_1}{2}$.

Proof. Expressing v in terms of theta function, we obtain

$$v := \frac{1}{q^{1/12}} \frac{(q^2;q^2)_\infty (q^3;q^3)_\infty}{(q;q)_\infty (q^6;q^6)_\infty}.$$

Now employing the definition of q-series and then taking logarithm on both sides and differentiating the resulting expresssion with respect to q, we deduce

$$\frac{q}{v}\frac{dv}{dq} = \sum_{n=1}^{\infty}\frac{6nq^{6n}}{1-q^{6n}} - \sum_{n=1}^{\infty}\frac{3nq^{3n}}{1-q^{3n}} - \sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}} + \sum_{n=1}^{\infty}\frac{nq^{n}}{1-q^{n}} - \frac{1}{12}.$$

Now. using the definition of Eisenstein series and the first level 6 identity of Lemma 2.4, we deduce the required differential equation.

Theorem 3.2. If

$$v := \frac{1}{q^{1/6}} \frac{f_1 f_3}{f_2 f_6}$$

then the following differential identity holds:

$$\frac{q}{v}\frac{dv}{dq} + \frac{1}{6}\sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_{k}\binom{n}{k}^2\binom{2k}{k}\right]X^n = 0,$$

where $X = \dfrac{\eta_1^2\eta_2^2\eta_3^2\eta_6^2}{z^2}$ and $z = \dfrac{6P_6 - 3P_3 + 2P_2 - P_1}{4}$.

Proof. Using the definition of theta function, q-series and then taking logarithm on both sides and differentiating the resulting relation with respect to q, we deduce

$$\frac{q}{v}\frac{dv}{dq}=\sum_{n=1}^{\infty}\frac{6nq^{6n}}{1-q^{6n}}-\sum_{n=1}^{\infty}\frac{3nq^{3n}}{1-q^{3n}}+\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}-\sum_{n=1}^{\infty}\frac{nq^{n}}{1-q^{n}}-\frac{1}{6}.$$

Employing the definition of Eisenstein series to the above relation and then using the second level 6 identity of Lemma 2.4, we deduce the required differential equation.

Theorem 3.3. If

$$v:=\frac{1}{q^{1/4}}\frac{f_1 f_2}{f_3 f_6}$$

then the following differential identity holds:

$$\frac{q}{v}\frac{dv}{dq}+\frac{1}{4}\sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_{k}\binom{n}{k}^{3}\right]X^{n}=0,$$

where $X=\dfrac{\eta_1^2\eta_2^2\eta_3^2\eta_6^2}{z^2}$ and $z=\dfrac{6P_6+3P_3-2P_2-P_1}{6}$.

Proof. Using the definition of theta function, q-series and then taking logarithm on both sides and differentiating, we derive

$$\frac{q}{v}\frac{dv}{dq}=\sum_{n=1}^{\infty}\frac{6nq^{6n}}{1-q^{6n}}+\sum_{n=1}^{\infty}\frac{3nq^{3n}}{1-q^{3n}}-\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}-\sum_{n=1}^{\infty}\frac{nq^{n}}{1-q^{n}}-\frac{1}{4}.$$

Employing the definition of Eisenstein series and then using the third level 6 identity of Lemma 2.4, we deduce the required differential equation.

Theorem 3.4 . If

$$v:=\frac{1}{q^{1/8}}\frac{f_2 f_4}{f_1 f_8}$$

then the following differential identity holds:

$$\frac{q}{v}\frac{dv}{dq}+\frac{1}{8}\sum_{n=0}^{\infty}\left[(-1)^{n}\binom{2n}{n}\sum_{k}\binom{n}{k}\binom{2k}{k}\binom{2n-k}{n-k}\right]X^{n}=0,$$

where $X=\dfrac{\eta_2^4\eta_4^4}{z^2}$ and $\left(\dfrac{\eta_1\eta_2\eta_5\eta_{10}}{z}\right)^{4/3}$ $z=\dfrac{8P_8-4P_4-2P_2+P_1}{3}$.

Proof. Employing the definition of theta function, q-series and then taking logarithm on both sides and differentiating the resulting expression with respect to q, we deduce

$$\frac{q}{v}\frac{dv}{dq}=\sum_{n=1}^{\infty}\frac{8nq^{8n}}{1-q^{8n}}-\sum_{n=1}^{\infty}\frac{4nq^{4n}}{1-q^{4n}}-\sum_{n=1}^{\infty}\frac{2nq^{2n}}{1-q^{2n}}+\sum_{n=1}^{\infty}\frac{nq^{n}}{1-q^{n}}-\frac{1}{8}.$$

Further, employing the definition of Eisenstein series and then using the level 8 identity of Lemma 2.4, we obtain the required differential equation.

Theorem 3.5. If

$$v:=\frac{1}{q^{1/6}}\frac{f_3^2}{f_1 f_9}$$

then the following differential identity holds:

$$\frac{q}{v}\frac{dv}{dq}+\frac{1}{6}\sum_{n=0}^{\infty}\left[\binom{2n}{n}\sum_{k}(-3)^{n-3k}\binom{n}{k}\binom{n-k}{k}\binom{n-2k}{k}\right]X^n=0,$$

Where $X=\frac{\eta_3^8}{z^2}$ and $z=\frac{9P_9-6P_3+P_1}{4}$.

Proof. Employing the definition of theta function, q-series and then taking logarithm on both sides and differentiating the resulting expression with respect to q, we deduce

$$\frac{q}{v}\frac{dv}{dq}=\sum_{n=1}^{\infty}\frac{9nq^{9n}}{1-q^{9n}}-2\sum_{n=1}^{\infty}\frac{3nq^{3n}}{1-q^{3n}}+\sum_{n=1}^{\infty}\frac{nq^{n}}{1-q^{n}}-\frac{1}{6}.$$

Now expressing the above relation in terms of Eisenstein series and then using the level 9 identity of Lemma 2.4, we deduce the required differential equation.

Theorem 3.6. If

$$v:=\frac{1}{q^{1/2}}\frac{f_1 f_2}{f_5 f_{10}}$$

then the following differential identity holds:

$$\frac{q}{v}\frac{dv}{dq}+\frac{1}{2}\sum_{n=0}^{\infty}\left[\sum_{k}\binom{n}{k}^4\right]X^n=0,$$

Where $X=\left(\frac{\eta_1\eta_2\eta_5\eta_{10}}{z}\right)^{4/3}$ and $z=\frac{10P_{10}+5P_5-2P_2-P_1}{12}$.

Proof. By using the definition of theta function, q-series and then taking logarithm on both sides and differentiating the resulting expression with respect to q, we deduce the expression in terms of Eisenstein series. Further, on employing the level 10 identity of Lemma 2.4, we deduce the required differential equation.

4. CONVOLUTION SUM

Definition 4.1. For $a,b \in \mathbb{N}$, the convolution sum is defined by

$$U_{a,b}(m) := \sum_{ai+bj=m} \sigma(i)\sigma(j).$$

where $a \le b$ and for any $l,m \in \mathbb{N}$, $\sigma_l(m) = \sum_{u/m} u^l$, and $\sigma_l(m) = 0$ for $m \notin \mathbb{N}$. For every nonnegative *m*, the convolution sum $\sum_{r+ks=m} \sigma(r)\sigma(s)$ has been assessed explicitly by A. Alaca et. al [6,7] and K. S. Williams [8]. Also E. X. W. Xia and O. X. M. Yao [9] have determined the illustrations for $\sum_{r+6s=m} \sigma(r)\sigma(s)$ and $\sum_{r+12s=m} \sigma(r)\sigma(s)$. Our proofs are simple, elementary and keys to our proofs are the claims of J. W. L. Glaisher [5],

$$P^2(q) = 1 + \sum_{l=1}^{\infty} (240\sigma_3(l) - 288 l\sigma_1(l))q^l. \qquad (2)$$

Theorem 4.2. For any $l \in \mathbb{N} - \{0\}$, the following identities hold:

$$i) \sum_{r+5s=l} \sigma(r)\sigma(s) = \frac{1}{24}\sigma_1(l) - \frac{1}{20} l\sigma_1(l) + \frac{1}{24}\sigma_3(l) + \frac{25}{24}\sigma_3\left(\frac{l}{5}\right) + \frac{1}{24}\sigma_1\left(\frac{l}{5}\right) - \frac{1}{4} l\sigma_1\left(\frac{l}{5}\right) - \frac{1}{360} A(l)$$

$$ii) \sum_{r+7s=l} \sigma(r)\sigma(s) = \frac{5}{168}\sigma_3(l) - \frac{1}{28} l\sigma_1(l) + \frac{35}{24}\sigma_3\left(\frac{l}{7}\right) - \frac{1}{4} l\sigma_1\left(\frac{l}{7}\right) + \frac{1}{24}\sigma_1(l) + \frac{1}{24}\sigma_1\left(\frac{l}{7}\right) - \frac{1}{224} B(l)$$

where $\sum_{l=1}^{\infty} A(l)q^l = \left[\sum_{l=1}^{\infty}\left[\binom{2l}{l}\sum_k \binom{l}{k}^2 \binom{l+k}{l}\right]X^l\right]^2$, $X = \frac{\eta_1^4\eta_5^4}{z^2}$, $z = \frac{5P_5 - P_1}{4}$

and $\sum_{l=1}^{\infty} B(l)q^l = \left[\sum_{l=1}^{\infty}\left[\sum_k \binom{l}{k}^2 \binom{2k}{l}\binom{l+k}{l}\right]X^l\right]^2$, $X = \left(\frac{\eta_1^2\eta_7^2}{z}\right)^{3/2}$, $z = \frac{7P_7 - P_1}{6}$.

Proof. i) On squaring the level 5 identity of Lemma 2.4, we get

$$P^2(q) + 25P^2(q^5) - 10P(q)P(q^5) = 16\left[\sum_{l=1}^{\infty}\left[\binom{2l}{l}\sum_k \binom{l}{k}^2 \binom{l+k}{l}\right]X^l\right]^2.$$

Now employing (2) and the definition of Eisenstein series and then equating the coefficients of q^l on either sides, we deduce (i).

ii) Similarly, squaring the level 7 identity of Lemma 2.4, employing (2) and the definition of Eisenstein series and then equating the coefficients of q^l on either sides, we obtain (ii).

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