Contour Map Study of Stresses in a Transversely Isotropic Elastic Plate due to Strip-Loading

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Abstract - *The stresses at any point of transversely isotropic elastic plate due to strip-loading have been obtained. Anti-plane strain problem with perfectly bonding boundary conditions is considered. Here, Fourier Transformation on equilibrium equations has been used to obtain the solution of the problem. As particular cases: the stresses in an isotropic elastic layered half-space and due to shear line-load in transversely isotropic elastic half space have been obtained. Also, the deformation of a transversely isotropic elastic half-space due to strip-loading can be obtained from our results. In order to study the effect of strip-loading in the elastic plate, contour maps have been presented.*

***Key Words*: Transversely isotropic material, Strip-loading, Shear line-load, Fourier transform, Perfectly bonded.**

1. INTRODUCTION

The solution of the problem of the deformation of a horizontally layered elastic material under the action of the surface loads has been finding wide applications in engineering, geophysics and soil mechanics. When the source surface is very long in one direction in comparison with the others, the use of two-dimensional approximation is justified and consequently calculations are simplified to a great extent and one gets a closed form analytical solution. A very long strip-source and a very long line-source are examples of such two-dimensional sources. Love [1] obtained expressions for the displacements due to a line source in an isotropic elastic medium. Maruyama [2] obtained the displacement and stress fields corresponding to long strike-slip faults in a homogenous isotropic half-space. Okada [3, 4] provided compact analytic expressions for the surface deformation and internal deformation due to inclined shear and tensile faults in homogenous isotropic half-space.

Garg et al. [5] obtained an analytical solution for the deformation of an orthotropic layered half-space caused by along strike-slip fault. Ting [6] derived the Green's functions for a line force and a screw dislocation for the anti-plane deformation of a monoclinic elastic medium consisting of a single half-space or two half-spaces in 'perfect' contact. Singh et al. [7] have studied the problem of two-dimensional static deformation of a monoclinic elastic medium using the eigenvalue method, following a Fourier transform. Madan et al. [8] have obtained the stresses in an Anisotropic Elastic Plate due to Strip-Loading. Madan et al. [9] have calculated the stresses in an orthotropic elastic layer lying over an irregular isotropic elastic half-space.

Lurding et al. [10] have taken an application of transversely isotropic materials to multi-layer shell elements undergoing finite rotations and large strains. Mallik and Kanoria[11] have discussed a two dimensional problem for a transversely isotropic generalized thermoelastic thick plate with spatially varying heat source. Shen et al. [12] have presented the fundamental solution for a two-dimensional displacement discontinuity method (DDM) for transversely isotropic elastic materials.

For a transversely isotropic material, there is an axis of material symmetry (defined as a direction with respect to which the material has identical properties) in addition to three planes of symmetry. The calculation of anti-plane deformation due to a line source in a transversely isotropic medium is much more difficult than the corresponding calculation for a source in an orthotropic medium because of the presence of the mixed derivatives in the equation of equilibrium. In geophysics, the elastic plate represents the crust of the earth.

In the present paper, the closed-form expressions for the stresses due to strip-loading in a horizontal transversely isotropic elastic plate of an infinite lateral extent lying over a base have been obtained. The interface between the plate and the base is considered to be 'perfectly bonded'. The deformation of the transversely isotropic elastic plate corresponding to 'perfectly bonded' boundary conditions has been obtained. The particular cases are considered as the stresses in an isotropic elastic layered half-space, the stresses due to shear line-load in transversely isotropic elastic half-space and the deformation of a transversely isotropic elastic uniform half-space due to strip-loading have been obtained. Contours have been presented to show the stress field causing the effect of strip-loading in a transversely isotropic elastic plate.

2. FUNDAMENTAL EQUATIONS

The constitutive matrix equation of a transversely isotropic material has the following form [13]

$$\begin{bmatrix}\tau_1\\ \tau_2\\ \tau_3\\ \tau_4\\ \tau_5\\ \tau_6\end{bmatrix}=\begin{bmatrix}c_{11} & c_{12} & c_{12} & 0 & 0 & 0\\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0\\ c_{12} & c_{23} & c_{22} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{2}(c_{22}-c_{23}) & 0 & 0\\ 0 & 0 & 0 & 0 & c_{66} & 0\\ 0 & 0 & 0 & 0 & 0 & c_{66}\end{bmatrix}\begin{bmatrix}e_1\\ e_2\\ e_3\\ e_4\\ e_5\\ e_6\end{bmatrix}. \qquad (1)$$

In equation (1), we used Voigt's convention by which the tensional indices are replaced by matrix indices in the expression of the stress and shear components τ_i and e_i $(i=1,2,3,4,5,6)$. The elements $c_{ij}, i, j=1,2,3,4,5,6$ of the stiffness matrix from (1) represent the elasticity's of the transversely isotropic material. A transversely isotropic material has 5 independent elastic constants and 12 non-zero terms.
The field's equations of a transversely isotropic material in anti-plane strain equilibrium state are:

-displacement equations:

$$u_1=u_1(x_2,x_3),\ u_2=u_3=0\,; \qquad (2)$$

-strain equations:

$$e_{11}=e_{22}=e_{33}=e_{23}=0,\ e_{12}=\frac{1}{2}u_{1,2},\ e_{13}=\frac{1}{2}u_{1,3}\,; \qquad (3)$$

-stress equations:

$$\tau_{11}=\tau_{22}=\tau_{33}=\tau_{23}=0,\ \ \tau_{12}=c_{66}\,u_{1,2},\ \tau_{13}=c_{66}u_{1,3}. \qquad (4)$$

Consequently, Cauchy's first equation yields the following, but second and third are identically satisfied

$$\tau_{12,2}+\tau_{13,3}=0\,. \qquad (5)$$

Using equations (4) and (5), the equilibrium equation satisfied by u_1 can be written in the following form:

$$u_{1,22}+u_{1,33}=0\,. \qquad (6)$$

3. FORMULATION AND SOLUTION OF THE PROBLEM

Consider a horizontal transversely isotropic elastic plate of thickness H lying over a base. The origin of Cartesian co-ordinates system $(x_1\, x_2\, x_3)$ is taken at the upper boundary of the plate and x_2-axis is drawn into the medium. The transversely isotropic elastic plate occupies the region $0<x_2\le H$ and the region $x_2>H$ is the base over which the plate is lying (Fig. 1).

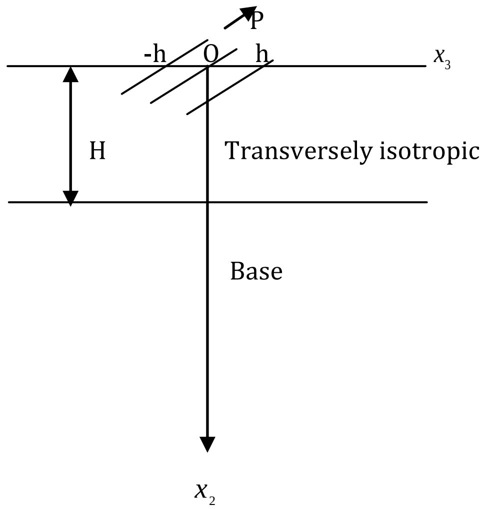

Fig -1: Section of the model by the plane x_1 = 0.

Let a shear-load P per unit area is acting over the strip $|x_3|\le h$ of the surface $x_2=0$ in the positive x_1-direction. The boundary condition at the surface $x_2=0$ is

$$\tau_{12}=\begin{cases}-P, & |x_3|\le h\\ 0, & |x_3|>h\end{cases} \qquad (7)$$

The interface $x_2=H$ between the plate and the base may be 'perfectly bonded'. Thus, at $x_2=H$, the continuity of the displacement and shear stress τ_{12} implies [14]

$$u_1(x_2=H-)=u_1(x_2=H+).$$

$$\tau_{12}(x_1=H-)=\tau_{12}(x_1=H+). \qquad (8)$$

We shall find the deformation field at any point of the transversely isotropic elastic plate corresponding to 'perfectly bonded' between the plate and the base due to strip-loading.

The Fourier transform of $X(x_2,x_3)$ is defined as

$$\overline{X}(x_2,k)=\int_{-\infty}^{\infty}X(x_2,x_3)e^{ikx_3}d\,x_3\,, \qquad (9)$$

so that

$$X(x_2,x_3)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\overline{X}(x_2,k)e^{-ikx_3}d\,k\,. \qquad (10)$$

Taking the Fourier transform of (6), we get

$$\frac{d^2\overline{u}_1}{d\,x_2^2}-k^2\,\overline{u}_1=0,\ \ c_{66}\neq 0. \qquad (11)$$

The solution of the ordinary differential equation (11) is

$$\bar{u}_1 = C_1 e^{|k| x_2} + C_2 e^{-|k| x_2}. \tag{12}$$

where C_1 and C_2 may be functions of *k*.

By using inverse Fourier transform, we have

$$u_1 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(C_1 e^{|k| x_2} + C_2 e^{-|k| x_2}\right) e^{-i k x_3} d k. \tag{13}$$

Using equation (13) and equation (4), the shear stresses are

$$\tau_{12} = \frac{c_{66}}{2\pi}\int_{-\infty}^{\infty}\left(C_1 e^{|k| x_2} - C_2 e^{-|k| x_2}\right) e^{-i k x_3} |k| d k, \tag{14}$$

$$\tau_{13} = -\frac{i c_{66}}{2\pi}\int_{-\infty}^{\infty}\left(C_1 e^{|k| x_2} + C_2 e^{-|k| x_2}\right) e^{-i k x_3} k d k \Bigg]. \tag{15}$$

Using the boundary condition (7), we have

$$\bar{\tau}_{12} = \frac{-2P}{\pi}\sin kh. \tag{16}$$

Therefore,

$$\tau_{12} = \frac{-P}{\pi}\int_{-\infty}^{\infty}\left(\frac{\sin kh}{k}\right) e^{-i k x_3} dk. \tag{17}$$

From (14) and (16), we obtain

$$C_1 - C_2 = -\frac{2P}{c_{66}}\left(\frac{\sin kh}{k|k|}\right). \tag{18}$$

3.1Perfectly Bonded

The displacement in the transversely isotropic elastic half-space $x_3 > H$ is

$$u_1' = \frac{1}{2\pi}\int_{-\infty}^{\infty} C_2' e^{-|k| x_2} e^{-i k x_3} dk, \tag{19}$$

where the coefficient C_2' is to be determined from the boundary conditions. Then

$$\tau_{12}' = -\frac{c_{66}'}{2\pi}\int_{-\infty}^{\infty} C_2' e^{-|k| x_2} e^{-i k x_3} |k| dk, \tag{20}$$

$$\tau_{13}' = -\frac{i c_{66}'}{2\pi}\int_{-\infty}^{\infty} C_2' e^{-|k| x_2} e^{-i k x_3} k dk. \tag{21}$$

Equations (8), (13), (14), (19) and (20) yield the relations

$$C_1 e^{|k|H} + C_2 e^{-|k|H} = C_2' e^{-|k|H} \tag{22}$$

$$c_{66}\left(C_1 e^{|k|H} - C_2 e^{-|k|H}\right) = -c_{66}' C_2' e^{-|k|H}. \tag{23}$$

Solving (18), (20) and (21), we get

$$\begin{aligned} C_1 &= \frac{2P\sin kh}{c_{66} k|k|}\left(\frac{m e^{-2|k|H}}{1 - m e^{-2|k|H}}\right), \\ C_2 &= \frac{2P\sin kh}{c_{66} k|k|}\left(\frac{1}{1 - m e^{-2|k|H}}\right), \\ C_2' &= \frac{4P\sin kh}{k|k|}\left(\frac{1}{1 - m e^{-2m_1|k|H}}\right)\left(\frac{1}{c_{66} + c_{66}'}\right), \end{aligned} \tag{24}$$

where $m = (c_{66} - c_{66}')/(c_{66} + c_{66}')$. (25)

Using (24) in equations (13), (14) and (15), the deformation field has been obtained as follow

$$u_1 = \frac{P}{\pi c_{66}}\int_{-\infty}^{\infty}\frac{\sin kh}{k|k|}\left[e^{-|k| x_2} + \sum_{n=1}^{\infty} m^n e^{-|k|(2nH + x_2)} + \sum_{n=1}^{\infty} m^n e^{-|k|(2nH - x_2)}\right] e^{-ik x_3} dk, \tag{26}$$

$$\tau_{12} = -\frac{P}{\pi}\left[\tan^{-1}\frac{2h x_2}{x_3^2 + x_2^2 - h^2} + \sum_{n=1}^{\infty} m^n\left\{\tan^{-1}\frac{2h(2nH + x_2)}{x_3^2 + (2nH + x_2)^2 - h^2}, - \tan^{-1}\frac{2h(2nH - x_2)}{x_3^2 + (2nH - x_2)^2 - h^2}\right\}\right] \tag{27}$$

$$\tau_{13} = \frac{P}{2\pi}\left[\log\frac{(x_3 + h)^2 + x_2^2}{(x_3 - h)^2 + x_2^2} + \sum_{n=1}^{\infty} m^n\left(\log\frac{(x_3 + h)^2 + (2nH + x_2)^2}{(x_3 - h)^2 + (2nH + x_2)^2} + \log\frac{(x_3 + h)^2 + (2nH - x_2)^2}{(x_3 - h)^2 + (2nH - x_2)^2}\right)\right] \tag{28}$$

for $0 \le x_2 \le H$ and for $x_2 > H$

$$u_1' = \frac{2P}{\pi}\left(\frac{1}{c_{66} + c_{66}'}\right)\int_{-\infty}^{\infty}\left(\frac{\sin kh}{k|k|}\right)\left[e^{-|k| x_2} + \sum_{n=1}^{\infty} m^n e^{-|k|(2nH + x_2)}\right] e^{-i k x_3} dk, \tag{29}$$

$$\tau'_{12} = -\frac{2P c_{66}'}{\pi(c_{66} + c_{66}')}\left[\tan^{-1}\frac{2h x_2}{x_3^2 + x_2^2 - h^2} + \sum_{n=1}^{\infty} m^n \tan^{-1}\frac{2h(2nH + x_2)}{x_3^2 + (2nH + x_2)^2 - h^2}\right], \tag{30}$$

$$\tau'_{13} = \frac{P}{\pi(c_{66} + c_{66}')}\left[\log\frac{(x_3 + h)^2 + x_2^2}{(x_3 - h)^2 + x_2^2} + \sum_{n=1}^{\infty} m^n \log\frac{(x_3 + h)^2 + (2nH + x_2)^2}{(x_3 - h)^2 + (2nH + x_2)^2}\right]. \tag{31}$$

4. PARTICULAR CASES

4.1. Isotropic Elastic Layered Half-Space

Taking $c_{66} = \mu$ in the equations (26)-(31), we obtain the deformation field for an isotropic elastic layered half-space

4.2.Shear Line-Load in Transversely isotropic Elastic Half Space

Taking $P = \frac{P_0}{2h}$ (shear Line-load) and proceeding to limit $h \to 0$, the deformation field caused by shear line-load P_0, per unit length, acting at the boundary $x_3 = 0$ of the semi-infinite transversely isotropic elastic medium in the positive x_1 − direction

$$u_1 = \frac{P_0}{\pi c_{66}} \int_0^{\infty} k^{-1} \cos x_3 \, e^{-k x_2} dk = -\frac{P_0}{2\pi c_{66}} \log\left[x_3^2 + x_2^2\right], \tag{32}$$

$$\tau_{12} = -\frac{P_0 \, x_2}{\pi\left[x_3^2 + x_2^2\right]} \tag{33}$$

$$\tau_{13} = -\frac{P_0 \, x_3}{\pi\left[x_3^2 + x_2^2\right]}. \tag{34}$$

4.3.Transversely isotropic elastic half-space due to strip-loading:

On taking $c_{66} = c'_{66}$ and $m = 0$ in the equations (29)-(31)

$$u_1 = \frac{P}{\pi c_{66}} \int_{-\infty}^{\infty} \frac{\sin kh}{k|k|} e^{-|k|x_2} e^{-i k x_3} dk., \tag{35}$$

$$\tau_{12} = -\frac{P}{\pi}\left[\tan^{-1} \frac{2hx_2}{x_3^2 + x_2^2 - h^2}\right], \tag{36}$$

$$\tau_{13} = \frac{P}{2\pi}\left[\log \frac{x_3^2 + x_2^2}{(x_3 - h)^2 + x_2^2}\right]. \tag{37}$$

5. NUMERICAL RESULTS AND CONCLUSIONS

In this section, the effect of strip-loading in transversely isotropic elastic plate has been studied. For this contour maps of stresses given in equations (27)-(28) and (30)-(31) have been presented. We use the values of elastic constants of Zinc Material

$c_{11} = 161,\ c_{22} = 61,\ c_{66} = 38.3,\ c_{12} = 34.2,\ c_{22} = 50.1$ (in Gpa)

for the layered medium and magnesium material with elastic constants

$c'_{11} = 59.7,\ c'_{22} = 61.7,\ c'_{66} = 16.4,\ c'_{12} = 26.2,\ c'_{22} = 21.7$ (in GPa)

for the transversely isotropic half-space (base) which are given by Bower[14] .

Define dimensionless distances, displacement and stresses as

$$X_2 = \frac{x_2}{H}, X_3 = \frac{x_3}{H}, \sigma_{12} = \frac{\tau_{12}}{H}, \sigma_{13} = \frac{\tau_{13}}{H}, \sigma'_{12} = \frac{\tau'_{12}}{H} \text{ and } \sigma'_{13} = \frac{\tau'_{13}}{H}.$$

Figures 2(a)-(d) and 3(a)-(d) show the variations of shear stresses σ_{12} and σ_{13}, respectively, for h=0.25, 0.50, 0.75 & 1 in the transversely isotropic elastic layer. Figures 4(a)-(d) and 5(a)-(d) present the variation of shear stresses σ_{12} and σ_{13}, respectively, for h=0.25, 0.50, 0.75 & 1 in the transversely isotropic elastic half-space. These contour analyses the stress field of transversely isotropic elastic plate corresponding to 'perfectly bonded' between the plate and the base due to strip-loading.

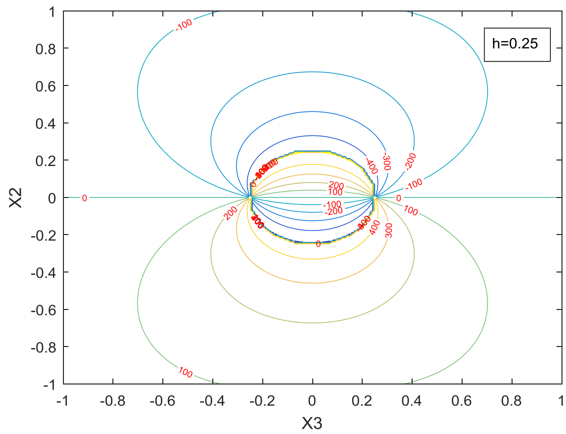

Fig. 2(a)

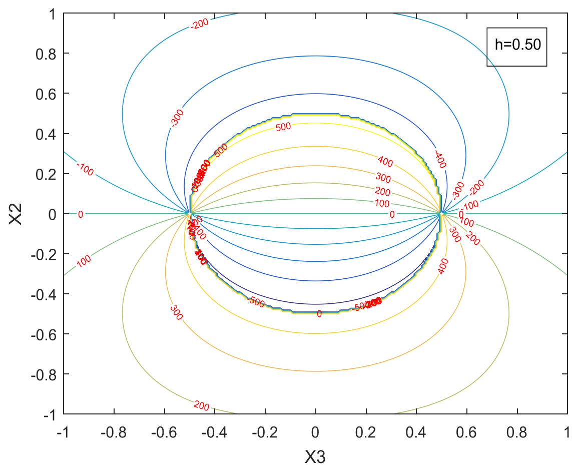

Fig. 2(b)

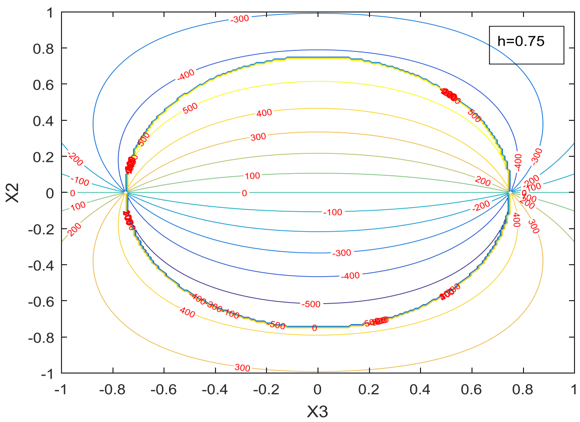

Fig. 2(c)

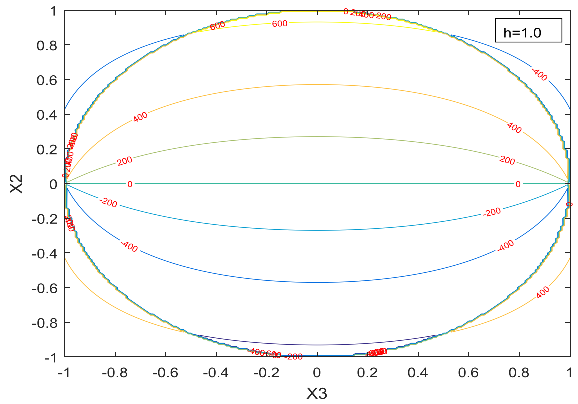

Fig. 2(d)

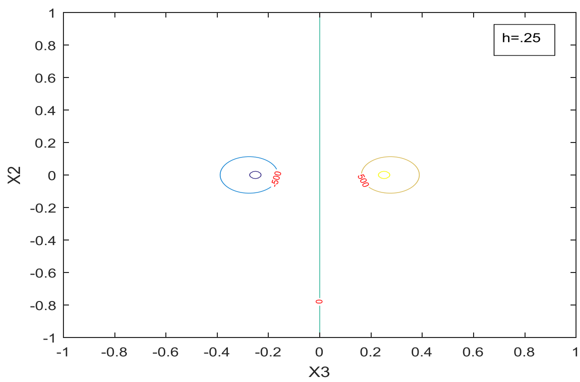

Fig. 3(a)

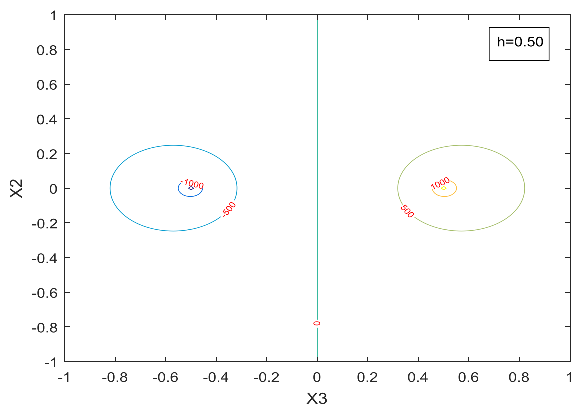

Fig. 3(b)

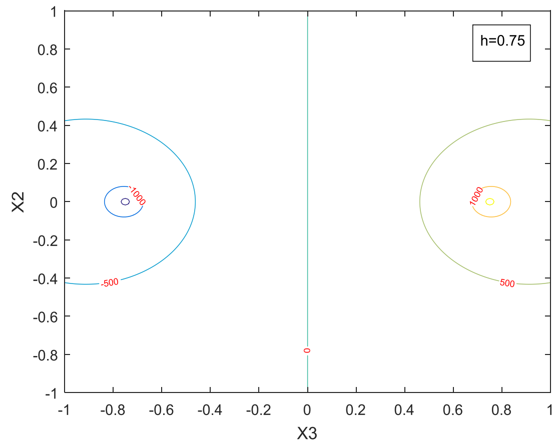

Fig.3(c)

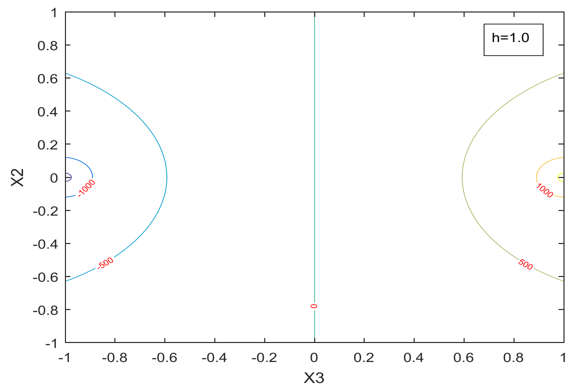

Fig. 3(d)

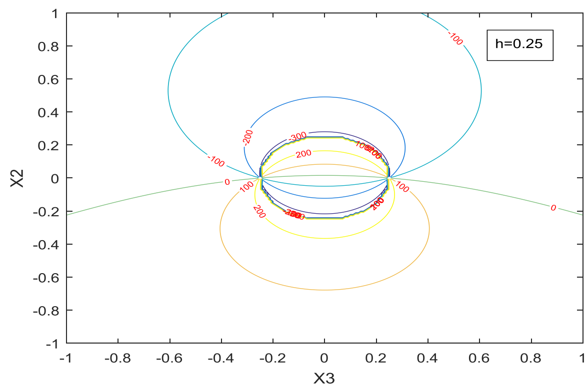

Fig. 4(a)

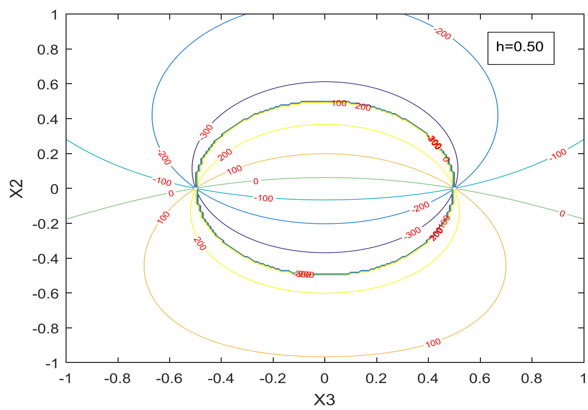

Fig. 4(b)

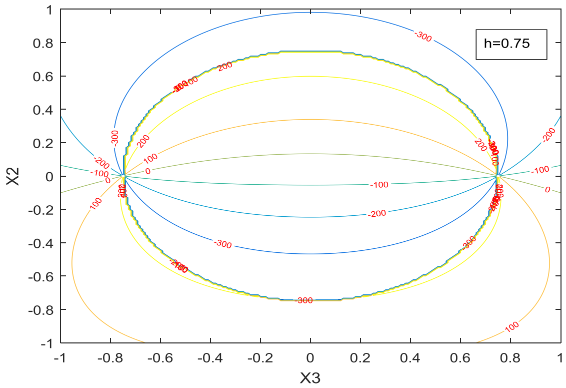

Fig. 4(c)

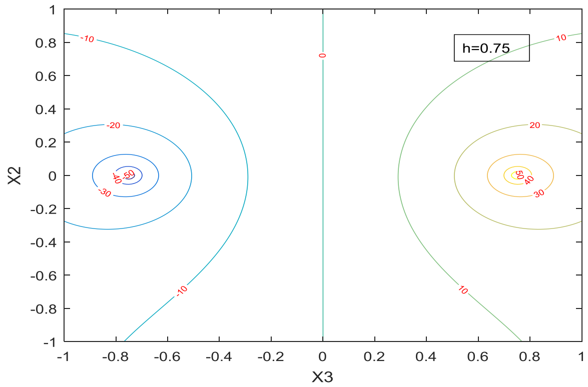

Fig. 5(c)

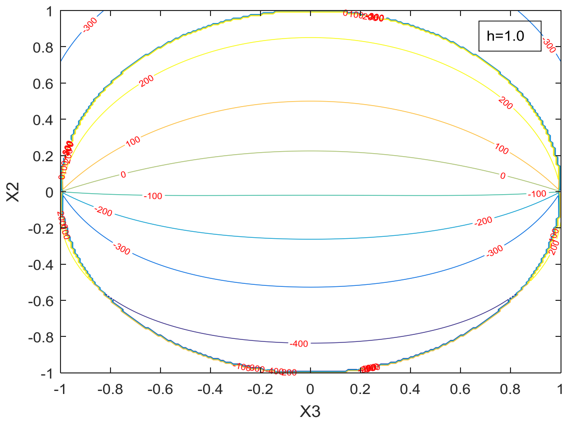

Fig. 4(d)

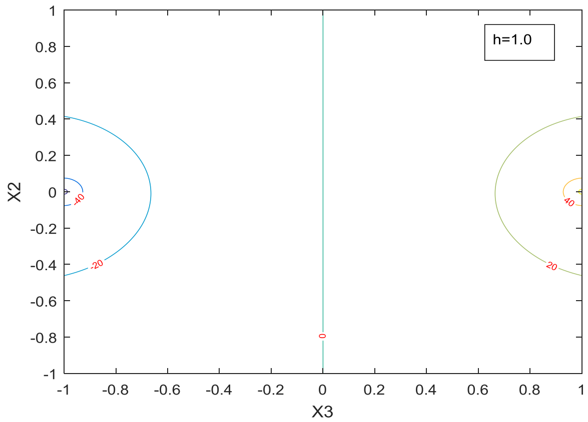

Fig. 5(d)

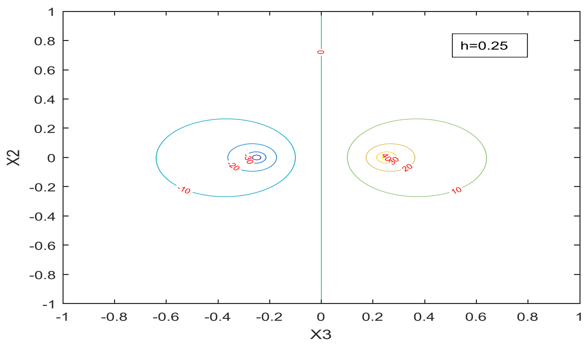

Fig. 5(a)

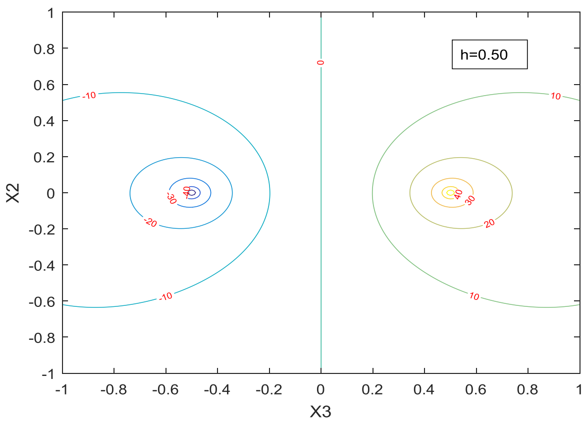

Fig. 5(b)

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