BUCKLING ANALYSIS OF FUNCTIONALLY GRADED MATERIAL PLATES UNDER THERMAL LOADING

Aarabhi K J[1], Anjana Susan John[2]

[1]PG Scholar, Department of Civil Engineering, SSET, Kerala, India
[2]Assistant Professor, Department of Civil Engineering, SSET, Kerala, India

---

Abstract - In this paper the analytical formulations and solutions using Generalized Shear Deformation Theory with five degrees of freedom for the buckling analysis of simply supported Sandwich Functionally Graded plates under various types of thermal loads are presented. Variation of material properties along the plate thickness is assumed to follow the Power Law function (Poisson's ratio is assumed to be constant for both the materials (metal and ceramic)). The Principle of Minimum Potential Energy (PMPE) is used to derive the equations of equilibrium. The analytical solutions in closed-form are obtained by solving the boundary value problem using Navier's Solution technique. For the analysis of plate problems, the simply supported boundary condition is considered. The numerical outcomes are acquired from a code created in MATLAB software.

***Key Words*: Functionally graded materials, generalized shear deformation theory, power law index, minimum potential energy, thermal buckling, Navier's solution**

1. INTRODUCTION

Sandwich structures are broadly utilized in territories of aircraft, aerospace, naval/marine, construction, transportation, and wind energy systems because of their outstanding properties, such as high stiffness and low weight for a long time. These sandwich structures offer advantages over other types of structures. But the sudden change in material properties over the interfaces between the face sheets and the core can bring about huge inter-laminar stresses which may cause delamination and this is a main problem of conventional sandwich structures. Also the difference in thermal coefficients of the materials may cause residual stresses. To overcome this problem the functionally graded materials (FGM) were introduced. FGM is heterogeneous composite material in which material properties vary continually from one surface to the other. This may be achieved by gradual change in the volume fraction of the constituent materials, mainly in the thickness direction and thus eliminating the sudden changes of thermo mechanical properties. This removes interfacial problems of composite materials; hence the stress distributions become smooth.

Two distinct material components were changed gradually from one another in simplest FGM form. Stepwise variation of the constituent materials resulting in discontinuity can also be treated as FGM. The most common FGM is ceramic to a metal gradation. Generally, FGMs are made from a metal and ceramic mixture or a combination of various materials. The ceramic part provides heat shield effects and protects the metal from oxidation and corrosion, and the metallic part toughens and strengthens the composition. Generally FGMs are used as composite structures exposed to high temperature conditions and various applications. FGM materials are having certain characteristic property changes continuously in space either in thickness or in- plane direction. Thus it is overcoming the shortcoming of traditional composite materials.

1.1 Material Property Idealization

FGM is formed by gradually varying the material composition. This can be achieved by varying the volume fraction distribution of component materials continuously and varying the mechanical and thermal properties simultaneously. The mathematical idealization of this non-homogeneous material property is done by assuming the FGM specimen as homogeneous, and defining the mechanical property variation according to corresponding relations. The material properties and volume fraction variations are calculated by different methods.

1.1.1 Power law function (P-FGM)

This method is the simplest and the one which is extensively used in literature. In this method, material properties and volume fraction vary across the thickness. The variation of material properties across the thickness (from metal surface to ceramic surface) is given by,

$$E_m z = E_m + \left(E_c - E_m\right)\left(V_f\right)^p$$

$$\alpha_m z = \alpha_m + \left(\alpha_c - \alpha_m\right)\left(V_f\right)^p$$

$$V_f = \frac{h}{z} + \frac{1}{2}$$

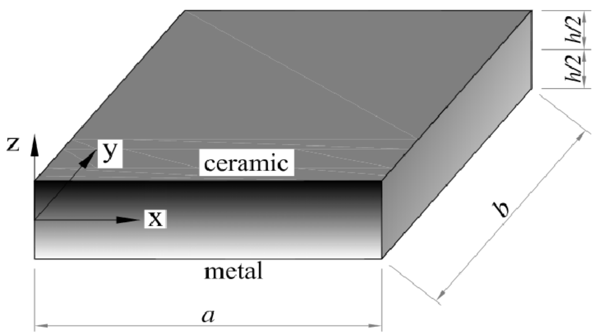

Figure 1.1 : Representation of Plate geometry

Where,

z is the axis in thickness direction.

E and α are Young's modulus and coefficient of thermal expansion.

h is the total thickness of the plate.

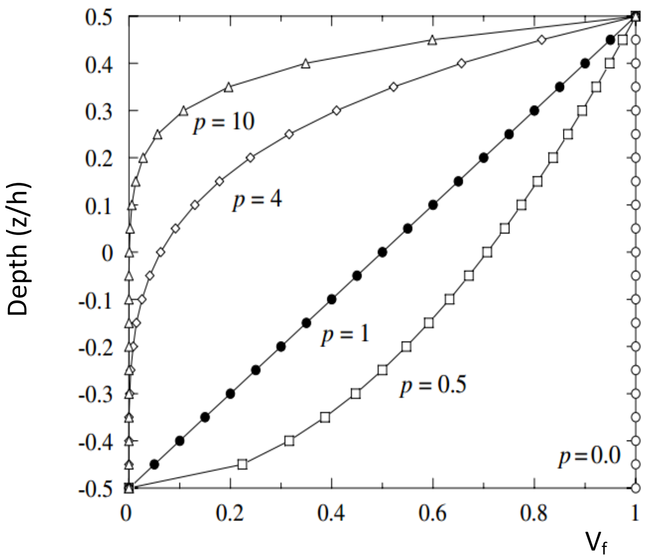

Figure 1.2 : Variation of Volume fraction V_f

V_f is the volume fraction of the ceramic surface, and p is power law index or material property gradient index. The subscripts c and m represent the components of ceramic and metal, respectively.

At bottom layer, z = -h/2, $E_z = E_m$

At top layer, z = h/2, $E_z = E_c$

1.2 Temperature Dependent Property

FGM are new composite materials which have been researched and developed for the parts that need to be temperature resistant. The temperature effect has significant influence on behaviour of FGM. A new method for evaluating the temperature dependency of FGM was proposed by Touloukian (1967)

$$P(T) = P_0\left(\frac{P_{-1}}{T} + 1 + P_1T + P_2T^2 + P_3T^3\right)$$

Where

P(T) is the relevant material property at temperature T.

P_{-1}, P_0, P_1, P_2 and P_3 are constants in the cubic fit of the material property and temperature.

The effective property that depends on both temperature and position is expressed as,

$$p_{eff}(z,T) = P_m(T) + \left(P_c(T) - P_m(T)\right)V_f$$

Where V_f is the volume fraction using power-law function and independent of temperature.

1.3 Thermal Considerations

Mechanical stress analysis alone is not sufficient for analysis of laminates that have been used at temperature different from the design operating temperature. In such cases, thermal stresses arise and must be accounted for. Variations in temperature results in two effects which are to be considered. First is, most material expands on heating and contracts on cooling, and this expansion is proportional to temperature change in most cases. Strain due to temperature change is added to the strains due to mechanical loading. Thermal strain can be expressed as

$$\varepsilon_T = \varepsilon \times \Delta T$$

Where,

α is the coefficient of thermal expansion.

ΔT is the temperature difference from reference state.

The second relates to strength and stiffness. On heating most materials become soft, weak and ductile. For stress analysis, the strength and the modulus of elasticity of the material at a temperature at which the structure is anticipated to perform in acquiring the natural frequency, or determining the buckling load of a cooled or heated structure is used. In an orthotropic material like composite, three different coefficients of thermal expansion and thermal strains can occur in all the orthogonal direction. There are no thermal effects in shear.

2. THEORETICAL FORMULATION

An all-side simply supported configuration is selected for the analysis. The displacement field assumed in sinusoidal shear deformation theory proposed by Zenkour is adopted. For buckling analysis, equilibrium equations are obtained using Principle of minimum potential energy. For simplicity, material properties; Poisson's ratio, coefficient of thermal expansion and Young's modulus are assumed to be temperature independent.

2.1 Geometry of sandwich plates

The sandwich plate is made of three isotropic layers of thickness h, width of band length a .

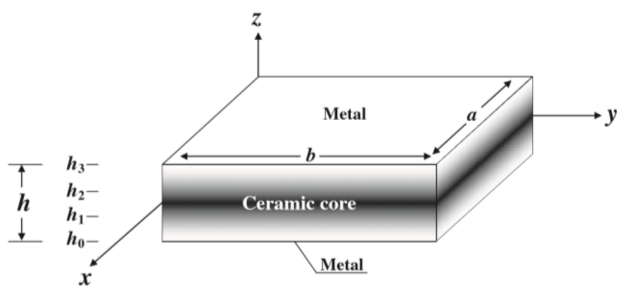

Figure 2.1 : Geometry of the FGM sandwich plate

The four edges of the FGM sandwich plate are simply supported. It is defined in the (x,y,z) coordinate system with x- and y-axes located in the middle plane (z= 0) with origin p at the corner of the plate. The external boundaries of the plate are defined by $z = \pm(h/2)$. The vertical positions of the bottom plane, both the interfaces between core and faces parts, and top plane are respectively denoted by $h_0 = -h/2,\ h_1,\ h_2$ *and* $h_3 = h/2$. The face layers are composed of FGM in which material properties vary evenly in the thickness direction (z) only. The FGM are made of metal and ceramic mixture and the core is completely ceramic. It is assumed that the gradation is varied from the interfaces to the bottom and top planes, i.e. bottom $(z = -h/2)$ & top $(z = +h/2)$ surfaces are metal-rich and the interfaces (h_1, h_2) are ceramic-rich. The thermal and mechanical properties of FGMs are computed from the volume fraction of the material composition. Poisson's ratio m is assumed to be constant, and the other material properties for every layer n, like the Young's modulus $E_{(n)}$ and coefficient of thermal expansion $\alpha_{(n)}$ at a point are generally assumed according to power law.

$$E_{(n)}(z) = E_m + (E_c - E_m)(V_{(n)})^p$$

$$\alpha_{(n)}(z) = \alpha_m + (\alpha_c - \alpha_m)(V_{(n)})^p$$

where subscripts m and c represent metal and ceramic respectively, and $V_{(n)}$ represents volume fraction of nth layer and its value is equal to unity in the core (i.e. $V_{(2)} = 1$ at $h_1 \le z \le h_2$)while it follows a simple power law through-the-thickness of the bottom and top layers that takes the form;

$$V_{(1)} = \left(\frac{z - h_0}{h_1 - h_0}\right), \quad h_0 \le z \le h_1$$

$$V_{(3)} = \left(\frac{z - h_3}{h_2 - h_3}\right), \quad h_2 \le z \le h_3$$

Where p refers to the parameter which indicates the power-law index whose values are greater than or equal to zero. The core is independent of p value as it is fully ceramic. The p value with zero indicates a homogeneous isotropic ceramic plate and the value with infinity indicates a metal–ceramic–metal (m–c–m) sandwich plate.

2.2 Strain Displacement Relations

The relation of the strains at whichever point within the plate and the corresponding deformations are functions of the assumed displacement fields with the definitions of strains for the linear theory of elasticity. The general linear strain-displacements relations are given as follows [Timoshenko and Goodier 1982]

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z},$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$$

The six quantities, i.e. three unit elongations in x, y, z directions $(\varepsilon_x, \varepsilon_y, \varepsilon_z)$, and three unit shear strains $(\gamma_{xy}, \gamma_{yz}, \gamma_{xz})$ in three orthogonal planes, are called components of strain at a point.

The stress-strain relations according to the general theory of elasticity are:

$$\sigma_x = \frac{E(z)}{1-\nu^2}\left[\varepsilon_x + \nu\,\varepsilon_y - (1+\nu)\alpha T\right]$$

$$, \sigma_y = \frac{E(z)}{1-\nu^2}\left[\nu\,\varepsilon_x + \varepsilon_y - (1+\nu)\alpha T\right],$$

$$\tau_{xy} = \frac{E(z)}{2(1+\nu)}\gamma_{xy}\ \tau_{yz} = \frac{E(z)}{2(1+\nu)}\gamma_{yz},\ \tau_{xz} = \frac{E(z)}{2(1+\nu)}\gamma_{xz}$$

For FGM the Q_{ij} matrix is given as follows,

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{xz}\end{bmatrix} = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13} & 0 & 0 & 0\\ Q_{21} & Q_{22} & Q_{23} & 0 & 0 & 0\\ Q_{31} & Q_{32} & Q_{33} & 0 & 0 & 0\\ 0 & 0 & 0 & Q_{44} & 0 & 0\\ 0 & 0 & 0 & 0 & Q_{55} & 0\\ 0 & 0 & 0 & 0 & 0 & Q_{66}\end{bmatrix} \times \begin{bmatrix}\varepsilon_x - \alpha T\\ \varepsilon_y - \alpha T\\ \varepsilon_z = 0\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{xz}\end{bmatrix}$$

Where $Q_{11} = Q_{22} = \frac{E(z)}{1-\nu^2}, \quad Q_{12} = Q_{21} = \frac{\nu E(z)}{1-\nu^2},$

$$Q_{44} = Q_{55} = Q_{66} = \frac{E(z)}{2(1+\nu)}$$

$$\begin{bmatrix}\sigma\\ \tau\end{bmatrix} = \begin{bmatrix} C_A & 0\\ 0 & G_A\end{bmatrix} \times \begin{bmatrix}\varepsilon_x - \varepsilon_T\\ \gamma\end{bmatrix}$$

Where,

$$C_A = \begin{bmatrix} Q_{11} & Q_{12} & Q_{13}\\ Q_{21} & Q_{22} & Q_{23}\\ Q_{31} & Q_{32} & Q_{33}\end{bmatrix} \qquad G_A = \begin{bmatrix} Q_{44} & 0 & 0\\ 0 & Q_{55} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}$$

$$\varepsilon_T = \alpha\ \Delta T(z)\begin{bmatrix}1\\1\\0\end{bmatrix}$$

In which α is the effective thermal coefficient and $\Delta T(z)$ is the temperature change which is given by $\Delta T(z) = T(z) - T_i$, where T_i and T(z) are the initial and the current temperatures respectively. And E(z) is determined by power-law function.

The stress resultant components for the FGM plate with thickness 'h' are given as:

$$\begin{bmatrix} N_x & N_x^* \\ N_y & N_y^* \\ N_{xy} & N_{xy}^* \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \begin{bmatrix} 1 & \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right) \end{bmatrix} dz$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} z\,dz$$

$$\begin{bmatrix} Q_x & Q_x^* \\ Q_y & Q_y^* \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} \begin{bmatrix} \tau_{xz} \\ \tau_{yz} \end{bmatrix} \begin{bmatrix} 1 & \cos\left(\frac{\pi z}{h}\right) \end{bmatrix} dz$$

$$N_x = \frac{1}{1-\nu^2}\left[D_1\overline{\varepsilon_{x_0}} + D_2\overline{k_x} + D_3\overline{k_x^*}\right] - \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\alpha(z)E(z)Tdz}{1-\nu}$$

$$N_y = \frac{1}{1-\nu^2}\left[D_1\overline{\varepsilon_{y_0}} + D_2\overline{k_y} + D_3\overline{k_y^*}\right] - \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\alpha(z)E(z)Tdz}{1-\nu}$$

$$M_x = \frac{1}{1-\nu^2}\left[D_2\overline{\varepsilon_{x_0}} + D_4\overline{k_x} + D_5\overline{k_x^*}\right] - \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{z\alpha(z)E(z)Tdz}{1-\nu}$$

$$M_y = \frac{1}{1-\nu^2}\left[D_2\overline{\varepsilon_{y_0}} + D_4\overline{k_y} + D_5\overline{k_y^*}\right] - \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{z\alpha(z)E(z)Tdz}{1-\nu}$$

$$N_x^* = \frac{1}{1-\nu^2}\left[D_3\overline{\varepsilon_{x_0}} + D_5\overline{k_x} + D_6\overline{k_x^*}\right] - \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\psi(z)\alpha(z)E(z)Tdz}{1-\nu}$$

$$N_y^* = \frac{1}{1-\nu^2}\left[D_3\overline{\varepsilon_{y_0}} + D_5\overline{k_y} + D_6\overline{k_y^*}\right] - \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{\psi(z)\alpha(z)E(z)Tdz}{1-\nu}$$

$$N_{xy} = \frac{1}{2(1+\nu)}\left[D_1\,\varepsilon_{xy_0} + D_2\,k_{xy} + D_3\,k_{xy}^*\right]$$

$$M_{xy} = \frac{1}{2(1+\nu)}\left[D_2\,\varepsilon_{xy_0} + D_4\,k_{xy} + D_5\,k_{xy}^*\right]$$

$$N_{xy}^* = \frac{1}{2(1+\nu)}\left[D_3\,\varepsilon_{xy_0} + D_5\,k_{xy} + D_6\,k_{xy}^*\right]$$

$$Q_x^* = \frac{D_7}{2(1+\nu)}\times\theta_x$$

$$Q_y^* = \frac{D_7}{2(1+\nu)}\times\theta_y$$

Where,

$$\overline{\varepsilon_{x_0}} = \varepsilon_{x_0} + \nu\,\varepsilon_{y_0}, \quad \overline{\varepsilon_{y_0}} = \varepsilon_{y_0} + \nu\,\varepsilon_{x_0}$$

$$\overline{k_x} = k_x + \nu\,k_y, \quad \overline{k_y} = k_y + \nu\,k_x$$

$$\overline{k_x^*} = k_x^* + \nu\,k_y^*, \quad \overline{k_y^*} = k_y^* + \nu\,k_x^*$$

$$D_1 = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\,dz \qquad D_2 = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\,z\,dz$$

$$D_3 = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)dz \qquad D_4 = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\,z^2\,dz$$

$$D_5 = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\,z\,\frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)dz \qquad D_6 = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\left[\frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right)\right]^2 dz$$

$$[D_7] = \int_{-\frac{h}{2}}^{\frac{h}{2}} E(z)\times\cos^2\left(\frac{\pi z}{h}\right)dz$$

$$\begin{bmatrix} N_T & M_T & N_T^* \end{bmatrix} = \int_{-\frac{h}{2}}^{\frac{h}{2}} C_A \begin{bmatrix} \alpha \\ \alpha \\ 0 \end{bmatrix} \begin{bmatrix} 1 & z & \frac{h}{\pi}\sin\left(\frac{\pi z}{h}\right) \end{bmatrix} \Delta T(z)$$

These thermal stress resultants are the functions of the incremental temperature $\Delta T(z)$

$$\begin{bmatrix} N_x \\ N_y \\ M_x \\ M_y \\ N_x^* \\ N_y^* \end{bmatrix} = \frac{1}{1-\nu^2}\begin{bmatrix} D_1 & 0 & D_2 & 0 & D_3 & 0 \\ 0 & D_1 & 0 & D_2 & 0 & D_3 \\ D_2 & 0 & D_4 & 0 & D_5 & 0 \\ 0 & D_2 & 0 & D_4 & 0 & D_5 \\ D_3 & 0 & D_5 & 0 & D_6 & 0 \\ 0 & D_3 & 0 & D_5 & 0 & D_6 \end{bmatrix} \times \begin{bmatrix} \overline{\varepsilon_{x_0}} \\ \overline{\varepsilon_{y_0}} \\ \overline{k_x} \\ \overline{k_y} \\ \overline{k_x^*} \\ \overline{k_y^*} \end{bmatrix} - \begin{bmatrix} N_T \\ N_T \\ M_T \\ M_T \\ N_T^* \\ N_T^* \end{bmatrix}$$

$$\begin{bmatrix} N \\ M \\ N^* \end{bmatrix} = \frac{1}{1-\nu^2}\begin{bmatrix} D_1 & D_2 & D_3 \\ D_2 & D_4 & D_5 \\ D_3 & D_5 & D_6 \end{bmatrix} \times \begin{bmatrix} \overline{\varepsilon} \\ \overline{k} \\ \overline{k^*} \end{bmatrix}$$

$$\begin{bmatrix} Q_x^* \\ Q_y^* \end{bmatrix} = \frac{1}{2(1+\nu)}\begin{bmatrix} D_7 & 0 \\ 0 & D_7 \end{bmatrix} \times \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix}$$

2.3 Equations of Equilibrium

The Principle of Minimum Potential Energy (PMPE) is used to derive the equations of equilibrium

The potential energy Π for a plate element is given by,

$$\Pi = U + V - W_{ex} - W_{ey}$$

Where,

U is the Strain energy of the plate due to deformation

V is the Potential energy due to the in-plane thermal stress

W_{ex} is Work done by edge stress on x (constant)

W_{ey} is Work done by edge stress on y (constant)

For Equilibrium, the total potential energy Π must be stationary. i.e.,

$$\delta\Pi = \delta(U + V - W_{ex} - W_{ey}) = 0$$

In analytical form Minimum Potential Energy (PMPE) can be expressed as:

$$\iiint_{x\,y\,z} \delta(U - W_{ex} - W_{ey})dV + \iiint_{x\,y\,z} \left(\sigma_x^{'}\delta\varepsilon_x^{'} + \sigma_y^{'}\delta\varepsilon_y^{'} + \tau_{xy}^{'}\delta\gamma_{xy}^{'}\right)dV = 0$$

The individual terms can be estimated as:

$$\delta U = \iiint_{x\,y\,z} (\sigma_x\,\delta\varepsilon_x + \sigma_y\,\delta\varepsilon_y + \sigma_z\,\delta\varepsilon_z + \tau_{xy}\,\delta\gamma_{xy} + \tau_{xz}\,\delta\gamma_{xz} + \tau_{yz}\,\delta\gamma_{yz})\,dx\,dy\,dz$$

The second term in the potential energy equation is the potential energy due to the in-plane thermal stresses $\sigma_x^{'}$ $\sigma_y^{'}$ and $\tau_{xy}^{'}$ produced by in-plane thermal moments and $\varepsilon_x^{'}, \varepsilon_y^{'}, \gamma_{xy}^{'}$ are in-plane strains due to transverse deflection w and are expressed as follows:

$$\varepsilon_x^{'} = -\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \qquad \varepsilon_y^{'} = -\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \qquad \gamma_{xy}^{'} = -\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)$$

$$\delta V = \iiint_{x\,y\,z}\left(\sigma_x^{'}\delta\varepsilon_x^{'} + \sigma_y^{'}\delta\varepsilon_y^{'} + \tau_{xy}^{'}\delta\gamma_{xy}^{'}\right)dxdydz$$

$$= -\frac{1}{2}\iiint_{x\,y\,z}\left[\sigma_x^{'}\delta\left(\frac{\partial w_0}{\partial x}\right)^2 + \sigma_y^{'}\delta\left(\frac{\partial w_0}{\partial y}\right)^2 + 2\tau_{xy}^{'}\delta\left(\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial w_0}{\partial y}\right)\right]dxdydz$$

$$= -\frac{1}{2}\iint_{x\,y}\left[N_{xT}\delta\left(\frac{\partial w_0}{\partial x}\right)^2 + N_{yT}\delta\left(\frac{\partial w_0}{\partial y}\right)^2 + 2N_{xyT}\delta\left(\frac{\partial w_0}{\partial x}\right)\left(\frac{\partial w_0}{\partial y}\right)\right]dxdy$$

Applying δ operation, the above equation becomes,

$$= -\frac{1}{2}\iint_{x\,y}\left[2N_{xT}\left(\frac{\partial w_0}{\partial x}\right)\partial\left(\frac{\partial w_0}{\partial x}\right) + 2N_{yT}\left(\frac{\partial w_0}{\partial y}\right)\partial\left(\frac{\partial w_0}{\partial y}\right) + 4N_{xyT}\left(\frac{\partial w_0}{\partial x}\right)\delta\left(\frac{\partial w_0}{\partial y}\right)\right]dxdy$$

$$\delta V = -\iint_{x\,y}\left[N_{xT}\left(\frac{\partial^2 w_0}{\partial x^2}\right)\delta w_0 + 2N_{xyT}\left(\frac{\partial^2 w_0}{\partial x\partial y}\right)\delta w_0 + N_{yT}\left(\frac{\partial^2 w}{\partial y^2}\right)\delta w_0\right]dxdy$$

$$\delta V = -\left[\iint N_{xT}\frac{\partial^2 w}{\partial x^2} + 2N_{xyT}\frac{\partial^2 w}{\partial x\partial y} + N_{yT}\frac{\partial^2 w}{\partial y^2}\right]\delta w dxdy$$

Where, $\begin{bmatrix}N_x & N_y & N_{xy}\end{bmatrix} = \int_{-h/2}^{h/2} [\sigma_x' \quad \sigma_y' \quad \tau_{xy}']\,dz$

W_{ex} and W_{ey} represents the work done by edge stresses on edges, x=constant and y= constant respectively.

$$W_{ex} = \frac{1}{2}\iint_{y\,z}(\overline{\sigma_x}u + \overline{\tau_{xy}}v + \overline{\tau_{xz}}w)dydz$$

$$W_{ey} = \frac{1}{2}\iint_{y\,z}(\overline{\sigma_y}v + \overline{\tau_{xy}}u + \overline{\tau_{yz}}w)dzdx$$

Rewriting the equation yield the following:

$$\int_{-h/2}^{h/2}\left[\begin{array}{l}\iint_{x\,y}\sigma_x\delta\varepsilon_x + \sigma_y\delta\varepsilon_y + \sigma_z\delta\varepsilon_z + \tau_{xy}\delta\gamma_{xy} + \tau_{xz}\delta\gamma_{xz} + \tau_{yz}\delta\gamma_{yz}dxdydz \\ -\iint N_{xT}\frac{\partial^2 w}{\partial x^2} + 2N_{xyT}\frac{\partial^2 w}{\partial x\partial y} + N_{yT}\frac{\partial^2 w}{\partial y^2}\delta w dxdy\end{array}\right] = 0$$

Substituting the proper strain expressions and then integrating along the thickness gives the stress resultants and integrating the resulting expression by parts and substituting coefficients of $\delta u_0, \delta v_0, \delta w_0, \delta\theta_x$ and $\delta\theta_y$ equal to zero, the following expressions are obtained which constitute the equilibrium equations for buckling analysis.

$$\delta u_0 : \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$\delta v_0 : \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0$$

$$\delta w_0 : \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} - N_{xT}\frac{\partial^2 w}{\partial x^2} - 2N_{xyT}\frac{\partial^2 w}{\partial x\partial y} - N_{yT}\frac{\partial^2 w}{\partial y^2} = 0$$

$$\delta\theta_x : \frac{\partial N_x^*}{\partial x} + \frac{\partial N_{xy}^*}{\partial y} - Q_x^* = 0$$

$$\delta\theta_y : \frac{\partial N_y^*}{\partial y} + \frac{\partial N_{xy}^*}{\partial x} - Q_y^* = 0$$

3. NAVIER SOLUTION TECHNIQUE

The FGM plate structure considered in the present work is of simply supported boundary condition. Such supports imply the following boundary conditions:

At edges x = 0 and x = a;

$v_0 = 0 \quad w_0 = 0 \quad \theta_x = 0 \quad M_x = 0 \quad N_x = 0$

At edges y = 0 and y = b;

$u_0 = 0; \quad w_0 = 0 \quad \theta_y = 0 \quad M_y = 0 \quad N_y = 0$

The generalized displacement field is expanded in double Fourier series to satisfy the assumed boundary conditions and is written as,

$$u_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{0mn}\cos\lambda x\,\sin\beta y \qquad \theta_x = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\theta_{xmn}\cos\lambda x\,\sin\beta y$$

$$v_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{0mn}\sin\lambda x\,\cos\beta y \qquad \theta_y = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\theta_{ymn}\sin\lambda x\,\cos\beta y$$

$$w_0 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{0mn}\sin\lambda x\,\sin\beta y$$

where

$$\lambda = \frac{m\pi}{a}, \qquad \beta = \frac{n\pi}{b}$$

Now the expressions for the curvature and slopes are substituted in the stress resultant-mid plane strain relationship and the relation can be written as,

$$\begin{bmatrix} N_x \\ N_y \\ M_x \\ M_y \\ N_x^* \\ N_y^* \end{bmatrix} = \frac{1}{1-\nu^2}\begin{bmatrix} D_1 & 0 & D_2 & 0 & D_3 & 0 \\ 0 & D_1 & 0 & D_2 & 0 & D_3 \\ D_2 & 0 & D_4 & 0 & D_5 & 0 \\ 0 & D_2 & 0 & D_4 & 0 & D_5 \\ D_3 & 0 & D_5 & 0 & D_6 & 0 \\ 0 & D_3 & 0 & D_5 & 0 & D_6 \end{bmatrix} \times \begin{bmatrix} \bar{\varepsilon}_{x_0} \\ \varepsilon_{y_0} \\ \bar{k}_x \\ \bar{k}_y \\ \bar{k}_x^* \\ \bar{k}_y^* \end{bmatrix} - \begin{bmatrix} N_T \\ N_T \\ M_T \\ M_T \\ N_T^* \\ N_T^* \end{bmatrix}$$

$$\begin{bmatrix} N \\ M \\ N^* \end{bmatrix} = \frac{1}{1-\nu^2}\begin{bmatrix} D_1 & D_2 & D_3 \\ D_2 & D_4 & D_5 \\ D_3 & D_5 & D_6 \end{bmatrix} \times \begin{bmatrix} \bar{\varepsilon} \\ \bar{k} \\ \bar{k}^* \end{bmatrix}$$

$$\begin{bmatrix} Q_x^* \\ Q_y^* \end{bmatrix} = \frac{1}{2(1+\nu)}\begin{bmatrix} D_7 & 0 \\ 0 & D_7 \end{bmatrix} \times \begin{bmatrix} \theta_x \\ \theta_y \end{bmatrix}$$

To solve the equilibrium equations, Substitute into the first equilibrium equation:

$$\delta u \ : \frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0$$

$$: \frac{1}{1-\nu^2} D_1\left[\frac{\partial^2 u}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 u}{\partial y^2}\right] + \frac{1}{2(1-\nu)} D_1\left[\frac{\partial^2 v}{\partial x \partial y}\right] - \frac{1}{1-\nu^2} D_2\left[\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\right]$$
$$+ \frac{1}{1-\nu^2} D_3\left[\frac{\partial^2 \theta_x}{\partial x^2} + \frac{1-\nu}{2}\frac{\partial^2 \theta_x}{\partial y^2}\right] + \frac{1}{2(1-\nu)} D_3\left[\frac{\partial^2 \theta_y}{\partial x \partial y}\right] = 0$$

Substituting Navier's solution form:

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left\{\begin{array}{l} -D_1\left[\dfrac{2\lambda^2+(1-\nu)\beta^2}{2(1-\nu^2)}\right]U_{0mn} - D_1\left[\dfrac{\lambda\beta}{2(1-\nu)}\right]V_{0mn} \\ +D_2\left[\dfrac{\lambda(\lambda^2+\beta^2)}{1-\nu^2}\right]W_{0mn} + -D_3\left[\dfrac{2\lambda^2+(1-\nu)\beta^2}{2(1-\nu^2)}\right]\theta_{xmn} \\ +D_3\left[\dfrac{\lambda\beta}{2(1-\nu)}\right]\theta_{ymn} \end{array}\right\}\cos\lambda x \sin\beta y = 0$$

On simplification:

$$: -D_1\left[\frac{2\lambda^2+(1-\nu)\beta^2}{2(1-\nu^2)}\right]U_{0mn} - D_1\left[\frac{\lambda\beta}{2(1-\nu)}\right]V_{0mn} + D_2\left[\frac{\lambda(\lambda^2+\beta^2)}{1-\nu^2}\right]W_{0mn}$$
$$- D_3\left[\frac{2\lambda^2+(1-\nu)\beta^2}{2(1-\nu^2)}\right]\theta_{xmn} + D_3\left[\frac{\lambda\beta}{2(1-\nu)}\right]\theta_{ymn} = 0$$

After following the steps, the solution is obtained in the following form:

$$([X]_{5\times5} - \psi[G]_{5x5}) \times \begin{bmatrix} U_{0mn} \\ V_{0mn} \\ W_{0mn} \\ \theta_{xmn} \\ \theta_{ymn} \end{bmatrix} = 0$$

[G] represents the coefficient matrix due to in-plane thermal forces , also called geometric stiffness matrix. Thermal stress resultants for an assumed rise in temperature ΔT is used to compute the coefficients of geometric stiffness.

$$G_{33} = \lambda^2 N_{xT} + \beta^2 N_{yT}$$

Lowest Eigen value ψ for the problem and thermal stresses are determined.

The critical temperature T_{cr} of the plate is calculated from thermal stress

$$N_{xT} = N_{yT}$$

[X] is the coefficient matrix and its elements are given as:

$$X_{11} = -D_1\left[\frac{2\lambda^2+(1-\nu)\beta^2}{2(1-\nu^2)}\right]$$

$$X_{12} = -D_1\left[\frac{\lambda\beta}{2(1-\nu)}\right]$$

$$X_{13} = +D_2\left[\frac{\lambda(\lambda^2+\beta^2)}{1-\nu^2}\right]$$

$$X_{14} = -D_3\left[\frac{2\lambda^2+(1-\nu)\beta^2}{2(1-\nu^2)}\right]$$

$$X_{15} = -D_3\left[\frac{\lambda\beta}{2(1-\nu)}\right]$$

$$X_{21} = -D_1\left[\frac{\lambda\beta}{2(1-\nu)}\right]$$

$$X_{22} = -D_1\left[\frac{2\beta^2+(1-\nu)\lambda^2}{2(1-\nu^2)}\right]$$

$$X_{23} = +D_2\left[\frac{\beta(\lambda^2+\beta^2)}{1-\nu^2}\right]$$

$$X_{24} = -D_3\left[\frac{\lambda\beta}{2(1-\nu)}\right]$$

$$X_{25} = -D_3\left[\frac{2\beta^2+(1-\nu)\lambda^2}{2(1-\nu^2)}\right]$$

$$X_{31} = +\frac{1}{1-\nu^2}D_2\left[\lambda\left(\lambda^2+\beta^2\right)\right]$$

$$X_{32} = +\frac{1}{1-\nu^2}D_2\left[\left(\lambda^2+\beta^2\right)\beta\right]$$

$$X_{33} = -\frac{1}{1-\nu^2}D_4\left[\left(\lambda^2+\beta^2\right)^2\right]$$

$$X_{34} = +\frac{1}{1-\nu^2}D_5\left[\lambda\left(\lambda^2+\beta^2\right)\right]$$

$$X_{35} = +\frac{1}{1-\nu^2}D_5\left[\left(\lambda^2+\beta^2\right)\beta\right]$$

$$X_{41} = -D_3\left[\frac{2\lambda^2+(1-\nu)\beta^2}{2(1-\nu^2)}\right]$$

$$X_{42} = -D_3\left[\frac{\lambda\beta}{2(1-\nu)}\right]$$

$$X_{43} = +D_5\left[\frac{\lambda(\lambda^2+\beta^2)}{1-\nu^2}\right]$$

$$X_{44} = +\left(-D_6\left[\frac{2\lambda^2+(1-\nu)\beta^2}{2(1-\nu^2)}\right] - \frac{D_7}{2(1+\nu)}\right)$$

$$X_{45} = -D_6\left[\frac{\lambda\beta}{2(1-\nu)}\right]$$

$$X_{51} = -D_3\left[\frac{\lambda\beta}{2(1-\nu)}\right]$$

$$X_{52} = -D_3\left[\frac{2\beta^2+(1-\nu)\lambda^2}{2(1-\nu^2)}\right]$$

$$X_{53} = +D_5\left[\frac{\beta(\lambda^2+\beta^2)}{1-\nu^2}\right]$$

$$X_{54} = -D_6\left[\frac{\lambda\beta}{2(1-\nu)}\right]$$

$$X_{55} = \left(-D_6\left[\frac{2\beta^2+(1-\nu)\lambda^2}{2(1-\nu^2)}\right]-\frac{D_7}{2(1+\nu)}\right)$$

4. CONCLUSIONS

Analytical formulation and stability analysis of functionally graded sandwich plates under various types of thermal loading using Sinusoidal shear deformation theory with five degrees of freedom are presented. The gradation of the properties through the thickness is supposed to vary as per to Power Law function (only modulus of elasticity and coefficient of thermal expansion is varied and Poisson's ratio is assumed to be constant for metal and ceramic). Based on the investigation completed and numerical outcomes obtained, the following conclusions are made:

- The critical buckling temperature of homogeneous plates is found to depend only on the coefficient of thermal expansion.
- As the ceramic content of the sandwich plates increases by reducing the power law index, the buckling strength is seen to be enhanced.
- The critical buckling temperature T_{cr} for the FGM plates are normally higher than that of homogeneous plates. FGM plates have numerous points of interest as a temperature resistant material, yet it is necessary to check their strength due to buckling failure.
- Assuming Non-Linear temperature variation through thickness results in higher values of critical buckling temperature for FGM plates

REFERENCES

[1] Akbarzadeh, A. H., Abbasi, M et al. (2012). "Coupled thermoelasticity of function- ally graded plates based on the third-order shear deformation theory". *Thin-Walled Structures*, 53, 141–155.

[2] Atmane, H. A., Bedia, E. A.Et al. (2016). "On the Thermal Buckling of Simply Supported Rectangular Plates Made of a Sigmoid Functionally Graded Al / Al 2 O3 Based Material"*Mechanics of Solids*, 51.177–187.

[3] Bakora, A., and Tounsi, A.(2015) "Thermo-mechanical post-buckling behaviour of thick function- ally graded plates resting on elastic foundations." *Structural Engineering and Mechanics*, 56, 1, 85–106.

[4] Barati, M. R., Zenkour, A. M et al. (2016). "Thermo-mechanical buckling analysis of embedded Nano size FG plates in thermal environments via an inverse co-tangential theory". *Composite Structures*, 141, 203–212.

[5] Bodaghi, M., and Saidi, (2011). "A. R. Thermoelastic buckling behaviour of thick functionally graded rectangular plates". *Archive of Applied Mechanics*, 81, 11, 1555–1572.

[6] Bouazza, M., Tounsi, A et al.(2010). "Thermoelastic stability analysis of functionally graded plates: An analytical approach."*Computational Materials Science* 49, 4, 865–870.

[7] Bouazza, M., Zenkour, A. M et al. (2018). "Closed-from solutions for the thermal buckling analyses of advanced nanoplates according to a hyperbolic four-variable refined theory with small-scale effects."*ActaMechanica*, 229, 5, 2251–2265.

[8] Bouderba, B., Sid, M et al.(2016). "Thermal stability of functionally graded sandwich plates using a simple shear deformation theory."*Structural Engineering and Mechanics*, 58 397–422.

[9] Bouiadjra, M. B., Sid, M et al. (2012). "Thermal Buckling of Functionally Graded Plates According to a Four-Variable Refined Plate theory graded plates according to a four-variable Refined Plate Theory." *Journal of Thermal Stresses*, 35, 677–694

[10] Bouiadjra, R. B., Bedia, E. A. A.(2013). "Nonlinear thermal buckling behaviour of functionally graded plates using an efficient sinusoidal shear deformation theory." 547–567.

[11] Bourada, M., Tounsi, A, E. (2012)."A new four-variable refined plate theory for thermal buckling analysis of functionally graded sandwich plates."*Journal of Sandwich Structures and Materials.* 14(1):5-33.

[12] Cheng, Z., &Bhatra, R. (2000). "Exact correspondence between eigenvalues", *Journal of sound and vibration*, 229(4), 879-895.

[13] Cˇukanovic´, D., Radakovicet al.(2017)."Thermal buckling analysis of metal-ceramic functionally graded plate according to high order shear deformation theory."*Journal of the Balkan Tribological Association*, 23, 3, 413–430.

[14] Daikh, A. A., and Megueni, A.(2017)."Thermal buckling analysis of functionally graded sandwich plates."*Journal of Thermal Stresses*, 41, 2, 1–21.

[15] Duc, N. D., and Van Tung, H. (2011). "Mechanical and thermal post-buckling of higher order shear deformable functionally graded plates on elastic foundations."*Composite Structures*, 93, 11, 2874–2881.

[16] Eslami, Y. K. M. R. (2008).“Thermal buckling and the post-buckling response of imperfect temperature-dependent sandwich FGM plates resting on two-parameter elastic foundations.”*Journal of applied mechanics*, 82,891–905.

[17] Fazzolari, F. A., and Carrera, E. (2014). “Thermal stability of FGM sandwich plates under various through-the-thickness temperature distributions”. 1449–1481.

[18] Han, S. C., Park, W. T et al. (2015). “A four-variable refined plate theory for dynamic stability analysis of S-FGM plates based on physical neutral surface.”*Composite Structures*, 131, 1081–1089.

[19] Jaberzadeh, E., Azhari, M et al. (2013). “Thermal buckling of functionally graded skew and trapezoidal plates with different boundary conditions using the element-free Galerkin method.”*European Journal of Mechanics,Solids* 42, 18–26.

[20] Javaheri, R. (2012). “Thermal Buckling of Functionally Graded Plates Based on Higher Order Theory.” *Journal of Thermal Stress*. 25(7), 603–625.

[21] Javaheri, R., and Eslami, M. R. (2002). “Thermal buckling of functionally graded plates”. *AIAA Journal*, 40, 1, 162–169.

[22] Kiani, Y., Bagherizadeh, E et al.(2011).“Thermal buckling of clamped thin rectangular FGM plates resting on Pasternak elastic foundation (Three approximate analytical solutions).”*ZAMM Zeitschrift fur AngewandteMathematik und Mechanik*,91, 7, 581–593.

[23] Lanhe, W. (2004). “Thermal buckling of a simply supported moderately thick rectangular FGM plate.”*Composite Structures*,64, 2, 211–218.

[24] Mantari, J. L., Ramos, I. A. (2016). “A Unified Formulation for Laminated Composite and Sandwich Plates Subject to Thermal Load Using Various Plate Theories”. *International Journal of Applied Mechanics*,8, 8, 1–35.

[25] Materials, C, and Akavci, S. S.(2014).“Thermal Buckling Analysis of Function- ally Graded Plates on an Elastic Foundation According to a Hyperbolic Shear Deformation Theory”. *Mechanics of Composite Materials*,50, 2, 197–212.

[26] Matsunaga, H. (2009). “Thermal buckling of functionally graded plates according to a 2D higher-order deformation theory”. *Composite Structures*, 90, 1, 76–86.

[27] Naderi, A., and Saidi, A. R. (2011). “Exact solution for stability analysis of moderately thick functionally graded sector plates on elastic foundation.”*Composite Structures,* 93, 2, 629– 638.

[28] Neves, A. M. A., Ferreira, A. J. M. (2013). “Static, free vibration and buckling analysis of isotropic and sandwich functionally graded plates using a quasi-3D higher-order shear deformation theory and a meshless technique.”*Composites Part B: Engineering,* 44, 1, 657–674.

[29] SamsamShariat, B. A., and Eslami, M. R. (2006). “Thermal buckling of imperfect functionally graded plates*”. International Journal of Solids and Structures*, 43, 14-15, 4082–4096.

[30] Shariat, B. A., and Eslami, M. R.(2007).“Buckling of thick functionally graded plates under mechanical and thermal loads.”*Composite Structures*, 78, 3, 433–439.

[31] Sobhy, M. (2016). “An accurate shear deformation theory for vibration and buckling of FGM sandwich plates in the hygrothermal environment.” *International Journal of Mechanical Sciences*, 110 62–77.

[32] Sohn, K., & Kim, J. (2008). “Structural stability of functionally graded panels subjected to thermal loads”. *Composite structures*, (82)317-325.

[33] Tebboune, W., Merdjah, M (2014). “Thermoelastic buckling response of thick functionally graded plates”. *Journal of Applied Mechanics and Technical Physics,* 55, 5, 857–869.

[34] Tung, H. V., and Duc, N. D. (2010). “Nonlinear analysis of stability for functionally graded plates under mechanical and thermal loads.”*Composite Structures*, 92, 5, 1184–1191.

[35] Vel, S. S., and Batra, R. C. (2002). “Exact Solution for Thermoelastic Deformations of Functionally Graded Thick Rectangular Plates.”*AIAA Journal*, 40, 7, 1421–1433.

[36] Yaghoobi, H., and Fereidoon, A. (2014). “Composites: Part B Mechanical and thermal buckling analysis of functionally graded plates resting on elastic foundations: An assessment of a simple, refined nth-order shear deformation theory.”*Composites part b*, 62, 54– 64.

[37] Yaghoobi, H., and Torabi, M.(2013).“Exact solution for thermal buckling of functionally graded plates resting on elastic foundations with various boundary conditions.”*Journal of Thermal Stresses*, 36, 9, 869–894.

[38] Yaghoobi, H., and Yaghoobi, P. (2013). “Buckling analysis of sandwich plates with FGM face sheets resting on elastic foundation with various boundary

conditions: an analytical approach."*Meccanica*,48, 2019–2035.

[39] Zenkour, A. M., and Mashat, D. S. (2010). "Thermal buckling analysis of ceramic-metal functionally graded plates."*Natural Science*, 2, 9, 968–978.

[40] Zenkour, A. M., and Sobhy, M.(2010)."Thermal buckling of various types of FGM sandwich plates." *Composite Structures*, 93, 1, 93–102.

[41] Zenkour, A. M., and Sobhy, M.(2011)."Thermal buckling of functionally graded plates resting on elastic foundations using the trigonometric theory."*Journal of Thermal Stresses*, 34, 11, 1119–1138.

[42] Zenkour, A. M., and Sobhy, M. (2013)."Nonlocal elasticity theory for thermal buckling of nanoplates lying on Winkler-Pasternak elastic substrate medium."*Physica E: Low- Dimensional Systems and Nanostructures*,53 251–259.