A Study on Fractional RLC Circuit

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Abstract - *In this paper, we use the method of finding general solution of linear homogeneous second order fractional differential equation with constant coefficients, regarding the modified Riemann-Liouville fractional derivatives to study the fractional RLC circuit problem. The Mittag-Leffler function and a new multiplication of fractional functions play important roles in this article.*

Key Words: Linear homogeneous second order fractional differential equation with constant coefficients, Modified Riemann-Liouville fractional derivatives, Fractional RLC circuit, Mittag-Leffler function, New multiplication, Fractional functions.

1. INTRODUCTION

Fractional derivatives of non-integer orders [1-3] have wide applications in physics and mechanics [4-9]. The rule of fractional derivative is not unique till date. The definition of fractional derivative is given by many authors. The commonly used definition is the Riemann-Liouvellie (R-L) definition [10-13]. Other useful definition includes Caputo definition of fractional derivative (1967) [10-13]. Jumarie's modification of R-L fractional derivative is useful to avoid nonzero fractional derivative of a constant functions [14].

A fractional β-order RLC circuit is an electrical circuit consisting of a fractional resistor R_β, a fractional inductor L_β, and a fractional capacitor C_β, connected in series or in parallel, where $0<\beta\le 1$. Assume that $i_c(t^\beta)$ and $v_c(t^\beta)$ are the β-order current and voltage of fractional order capacitor respectively, then the model that involves both characteristics can be described by the following relationship, given in [20]

$$i_c(t^\beta)=C_\beta\left({}_0D_t^\beta\right)[v_c(t^\beta)]. \tag{1}$$

Similarly, assuming that $i_L(t^\beta)$ and $v_L(t^\beta)$ are the current and voltage of fractional order inductor respectively, then the model that involves the characteristics can be described by the following relationship, given in [20]

$$v_L(t^\beta)=L_\beta\left({}_0D_t^\beta\right)[i_L(t^\beta)]. \tag{2}$$

In circuit designs, the general theorems of fractional order oscillators and filters are introduced through analytical conditions, numerical analysis, circuit simulations, and experimental results [21-22]. The generalized fundamentals of the conventional LC tank circuit are presented in [23] showing new responses, which exist only in the fractional order case. In addition, the stability analysis of the fractional order RLC circuit is introduced in [24] for independent fractional-orders.

In this study, the method of seeking general solution of linear homogeneous second order fractional differential equation with constant coefficients, regarding the modified Riemann-Liouville fractional derivatives is used to solve the fractional RLC circuit problem. Furthermore, the Mittag-Leffler function and a new multiplication of fractional functions play important roles in this paper. On the other hand, our approach is different from [24-27], and it is the generalization of the method for solving classical RLC circuit problem. The source free fractional RLC filter can be described as a fractional second order circuit, meaning that any voltage or current in the circuit can be described by a fractional second order differential equation in circuit analysis. This can usefully be expressed in a more generally applicable form:

$$\left(\left({}_0D_t^\beta\right)^2+2\alpha\left({}_0D_t^\beta\right)+{\omega_0}^2\right)[x(t^\beta)]=0, \tag{3}$$

where α is the β-order neper frequency, or attenuation, ω_0 is the β-order resonance frequency. $\omega_0=\frac{1}{\sqrt{L_\beta C_\beta}}$ and $=\frac{1}{2R_\beta C_\beta}$, if the circuit is parallel; $\alpha=\frac{R_\beta}{2L_\beta}$, if the circuit is series.

2. PRELIMINARIES

In this section, we introduce the fractional differentiation and a new multiplication we used in this article and study their properties.

Notation 2.1: If β is a real number, then

$$[\beta]=\begin{cases}0, & \text{if } \beta<0,\\ \text{the greatest integer less than or equal to } \alpha, & \text{if } \beta\ge 0\end{cases}$$

Definition 2.2: Let β be a real number, m be a positive integer, and $f(t)\in C^{[\beta]}([a,b])$. The modified Riemann-Liouville fractional derivatives of Jumarie type ([14]) is defined by

$$\left({}_aD_t^\beta\right)[f(t)]$$
$$=\begin{cases}\dfrac{1}{\Gamma(-\beta)}\displaystyle\int_a^t (t-\tau)^{-\beta-1}f(\tau)d\tau, & \text{if } \beta<0\\ \dfrac{1}{\Gamma(1-\beta)}\dfrac{d}{dt}\displaystyle\int_a^t (t-\tau)^{-\beta}[f(\tau)-f(a)]\,d\tau, & \text{if } 0<\beta<1\\ \dfrac{d^m}{dt^m}\left({}_aD_t^{\beta-m}\right)[f(t)], & \text{if } m\le\beta<m+1\end{cases} \tag{4}$$

where $\Gamma(y)=\int_0^{\infty} x^{y-1}e^{-x}\,dx$ is the gamma function defined on $y>0$. For any positive integer n, we define $\left({}_aD_t^{\beta}\right)^n=\left({}_aD_t^{\beta}\right)\left({}_aD_t^{\beta}\right)\cdots\left({}_aD_t^{\beta}\right)$, the n -th order fractional derivative of ${}_aD_t^{\beta}$. We have the following properties.

Proposition 2.3 ([15]): *Suppose that β,γ,c are real constants and $0<\beta\le 1$, then*

$$\left({}_0D_t^{\beta}\right)[t^{\gamma}]=\frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\beta+1)}t^{\gamma-\beta},\quad \text{if } \gamma\ge\beta \tag{5}$$

$$\left({}_0D_t^{\beta}\right)[c]=0, \tag{6}$$

Definition 2.4 ([16]): If $\beta>0$, and z is a complex variable. The Mittag-Leffler function is defined by

$$E_{\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\beta+1)}. \tag{7}$$

Definition 2.5 ([17]): Let $0<\beta\le 1$, λ be a complex number, and t be a real variable, then $E_{\beta}(\lambda t^{\beta})$ is called β-order fractional exponential function, and the β-order fractional cosine and sine function are defined by

$$cos_{\beta}(\lambda t^{\beta})=\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k}t^{2k\beta}}{\Gamma(2k\beta+1)}, \tag{8}$$

and

$$sin_{\beta}(\lambda t^{\beta})=\sum_{k=0}^{\infty}\frac{(-1)^k\lambda^{2k+1}t^{(2k+1)\beta}}{\Gamma((2k+1)\beta+1)}. \tag{9}$$

Proposition 2.6 (fractional Euler's formula) ([18]): *Let $j=\sqrt{-1}$ and $0<\beta\le 1$, then*

$$E_{\beta}(jt^{\beta})=cos_{\beta}(t^{\beta})+j\,sin_{\beta}(t^{\beta}). \tag{10}$$

Next, we define a new multiplication of fractional functions such that some properties, for instance, product rule and chain rule are correct [19].

Definition 2.7 : Assume that λ,μ,z are complex numbers, $0<\beta\le 1$, m,l,k are non-negative integers, and a_k,b_k are real numbers, $p_k(z)=\frac{1}{\Gamma(k\beta+1)}z^k$ for all k. Then we define $p_m(\lambda t^{\beta})\otimes p_l(\mu s^{\beta})$

$$=\frac{1}{\Gamma(m\beta+1)}(\lambda t^{\beta})^m\otimes\frac{1}{\Gamma(l\beta+1)}(\mu s^{\beta})^l$$

$$=\frac{1}{\Gamma((m+l)\beta+1)}\binom{m+l}{m}(\lambda t^{\beta})^m(\mu s^{\beta})^l, \tag{11}$$

where $\binom{m+l}{m}=\frac{(m+l)!}{m!\,l!}$.

If $f_{\beta}(\lambda t^{\beta})$ and $g_{\beta}(\mu s^{\beta})$ are two fractional functions,

$$f_{\beta}(\lambda t^{\beta})=\sum_{k=0}^{\infty}a_k\,p_k(\lambda t^{\beta})=\sum_{k=0}^{\infty}\frac{a_k}{\Gamma(k\beta+1)}(\lambda t^{\beta})^k, \tag{12}$$

$$g_{\beta}(\mu s^{\beta})=\sum_{k=0}^{\infty}b_k\,p_k(\mu s^{\beta})=\sum_{k=0}^{\infty}\frac{b_k}{\Gamma(k\beta+1)}(\mu s^{\beta})^k, \tag{13}$$

then we define

$$f_{\beta}(\lambda t^{\beta})\otimes g_{\beta}(\mu s^{\beta})$$

$$=\sum_{k=0}^{\infty}a_k\,p_k(\lambda t^{\beta})\otimes\sum_{k=0}^{\infty}b_k\,p_k(\mu s^{\beta})$$

$$=\sum_{k=0}^{\infty}\left(\sum_{q=0}^{k}a_{k-q}b_q p_{k-q}(\lambda t^{\beta})\otimes p_q(\mu s^{\beta})\right). \tag{14}$$

Proposition 2.8 ([19]): $f_{\beta}(\lambda t^{\beta})\otimes g_{\beta}(\mu s^{\beta})$

$$=\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\beta+1)}\sum_{q=0}^{k}\binom{k}{q}a_{k-q}b_q\,(\lambda t^{\beta})^{k-q}(\mu s^{\beta})^q. \tag{15}$$

Remark 2.9: The $\otimes$ multiplication satisfies the commutative law and the associate law, and it is the generalization of traditional multiplication, since the $\otimes$ multiplication becomes the ordinary multiplication if $\beta=1$.

Proposition 2.10 : $E_{\beta}(\lambda t^{\beta})\otimes E_{\beta}(\mu s^{\beta})=E_{\beta}(\lambda t^{\beta}+\mu s^{\beta})$.

$$\tag{16}$$

Corollary 2.11: $E_{\beta}(\lambda t^{\beta})\otimes E_{\beta}(\mu t^{\beta})=E_{\beta}((\lambda+\mu)t^{\beta})$.

$$\tag{17}$$

The following is the major result we used in this paper to study the fractional RLC circuit.

Theorem 2.12: *Let $0<\beta\le 1$, $a,b,A,B,C,D_1,D_2,E_1,E_2,F_1,F_2$ be real constants, and $A\ne 0$. Let $x_h(t^{\beta})$ be the general solution of the linear homogeneous second order fractional differential equation with constant coefficients*

$$\left(A\left({}_0D_t^{\beta}\right)^2+B\left({}_0D_t^{\beta}\right)+C\right)\left[x(t^{\beta})\right]=0. \tag{18}$$

Suppose that s_1,s_2 are two roots of the characteristic equation of Eq. (18)

$$As^2+Bs+c=0 \tag{19}$$

Case 1. If s_1,s_2 are two distinct real numbers, then

$$x_h(t^{\beta})=D_1\,E_{\beta}(s_1t^{\beta})+D_2\,E_{\beta}(s_2t^{\beta}). \tag{20}$$

Case 2. If $s_1=s_2=s$ are the same real numbers, then

$$x_h(t^{\beta})=(E_1+E_2t^{\beta})\otimes E_{\beta}(st^{\beta}). \tag{21}$$

Case 3. If $s_1=a+jb$, $s_2=a-jb$ are conjugate complex numbers, then

$$x_h(t^{\beta})=E_{\beta}(at^{\beta})\otimes\left(F_1cos_{\beta}(bt^{\beta})+F_2sin_{\beta}(bt^{\beta})\right). \tag{22}$$

3. CIRCUIT ANALYSIS

3.1 Series Fractional RLC Circuit

Let $0<\beta\le 1$ and $i(t^{\beta})$ be the β-order fractional current. Then the source free series fractional RLC circuit satisfies

$$\left(\left({}_0D_t^{\beta}\right)^2+2\alpha\left({}_0D_t^{\beta}\right)+\omega_0{}^2\right)\left[i(t^{\beta})\right]=0, \tag{23}$$

where $\alpha=\frac{R_{\beta}}{2L_{\beta}}$ and $\omega_0=\frac{1}{\sqrt{L_{\beta}C_{\beta}}}$.

Theorem 3.1.1: *Consider the second order fractional differential equation* (23).

Case 1. If $\alpha > \omega_0$ *: overdamped response, then Eq.* (23) *has the general solution*

$$i(t^\beta) = D_1\, E_\beta(s_1 t^\beta) + D_2\, E_\beta(s_2 t^\beta), \qquad (24)$$

where $s_1 = -\alpha + \sqrt{\alpha^2 - {\omega_0}^2}$, $s_2 = -\alpha - \sqrt{\alpha^2 - {\omega_0}^2}$.

Case 2. If $\alpha = \omega_0$ *: critically damped response, then*

$$i(t^\beta) = (E_1 + E_2 t^\beta) \otimes E_\beta(-\alpha t^\beta). \qquad (25)$$

Case 3. If $\alpha < \omega_0$ *: underdamped response, then*

$$i(t^\beta) = E_\beta(-\alpha t^\beta) \otimes \left(F_1 cos_\beta(\omega_d t^\beta) + F_2 sin_\beta(\omega_d t^\beta)\right), \qquad (26)$$

where $\omega_d = \sqrt{{\omega_0}^2 - \alpha^2}$ *is the damped resonance frequency or the damped natural frequency.*

3.2 Parallel Fractional RLC Circuit

If $0 < \beta \leq 1$ and $v(t^\beta)$ is the β-order fractional voltage, then the source free parallel fractional RLC circuit satisfies

$$\left(\left({}_0D_t^\beta\right)^2 + 2\alpha\left({}_0D_t^\beta\right) + {\omega_0}^2\right)[v(t^\beta)] = 0, \qquad (27)$$

where $\alpha = \frac{1}{2R_\beta C_\beta}$ and $\omega_0 = \frac{1}{\sqrt{L_\beta C_\beta}}$.

Theorem 3.2.1: *Consider the second order fractional differential equation* (27).

Case 1. If $\alpha > \omega_0$ *: overdamped response, then Eq.* (27) *has the general solution*

$$v(t^\beta) = D_1\, E_\beta(s_1 t^\beta) + D_2\, E_\beta(s_2 t^\beta), \qquad (28)$$

where $s_1 = -\alpha + \sqrt{\alpha^2 - {\omega_0}^2}$, $s_2 = -\alpha - \sqrt{\alpha^2 - {\omega_0}^2}$.

Case 2. If $\alpha = \omega_0$ *: critically damped response, then*

$$v(t^\beta) = (E_1 + E_2 t^\beta) \otimes E_\beta(-\alpha t^\beta). \qquad (29)$$

Case 3. If $\alpha < \omega_0$ *: underdamped response, then*

$$v(t^\beta) = E_\beta(-\alpha t^\beta) \otimes \left(F_1 cos_\beta(\omega_d t^\beta) + F_2 sin_\beta(\omega_d t^\beta)\right), \qquad (30)$$

4. CONCLUSION

There are many different methods to deal with fractional RLC circuit problem. The approach we provided in this paper is the generalization of solving traditional RLC circuit problem. Therefore, the results we obtained are closely related with the classical results in RLC circuit. Moreover, our method can be extended to solve another physical problems. In the future, we will use the fractional differential techniques to study another engineering mathematics problems.

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