ANALATICAL AND NUMERICAL STUDY OF HEAT TRANSFER IN FINS

Dr. Osama Mohammed Elmardi Suleiman Khayal[1], PhD. Student Moataz Abdelgadir Ali Abdelgadir2, Dr. Salaheldin Hassabelgabo Abdelrazig Ibrahim3

[1,3]Department of Mechanical Engineering, Faculty of Engineering and Technology, Nile Valley University, Atbara – Sudan

[2]Faculty of Engineering Mechanical Engineering Department, Red Sea University-Sudan

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Abstract - *In this paper, one-dimensional heat transfer problems that can be approached by different methods was presented. The analytical approach using the direct solutions of differential equations was used to verify the approximate solution. In this paper the analysis was focused on the insulated fin tip. A more practical approach is the use of a suitable numerical method. The finite volume method seems to provide a good approach in using these complex problems with a variety of boundary conditions and the usage of MATLAB programming. An iterative technique may frequently yield a more efficient solution to the nodal equations than a direct matrix inversion. With the help of finite volume method and MATLAB programming for insulated and uniform cross-sectional area of the fin problems and with applying GAUSS-SEIDEL iteration method which is used to solve and calculate the nodal temperature distribution at different positions along the fin rod, it was found that the approximate solution is very close to the analytical solution.*

Key Words: Heat Transfer, Uniform Cross-sectional Area of the Fin, Finite Volume Method, Analytical Solution.

1. INTRODUCTION

A fin is a surface that extends from an object to increase the rate of heat transfer to or from the environment by increasing convection. For the principle of conduction, convection, radiation of an pin configuration determines the amount of heat it transfers increasing the temperature difference between the fin configuration and the depends on the environment, slightly increasing the convection heat transfer coefficient, or slightly increasing the surface area of the pin configuration of the object increases the heat transfer. Circumferential fins around the cylinder, square and rectangular shape of a motorcycle engine and fins attached to condenser tubes of a refrigerator are a few familiar examples only occurs when there is a temperature difference, flows faster when this difference is higher, always flows from high to low temperature, is greater with the greater surface area. Fins are most commonly used in heat-exchanging devices such as radiators in cars, computer CPU heatsinks, and heat exchangers in power plants. They are also used in newer technology such as hydrogen fuel cells. Nature has also taken advantage of the phenomena of fins. Fins are used in a large number of applications to increase the heat transfer from surfaces. Typically, the fin material has a high thermal conductivity. The surface area and can sometimes be an economical solution to heat transfer problems. Different fin configurations are illustrated in figure (1). A straight fin is any extended surface that is attached to a plain wall. It may be of uniform cross-sectional area, or its cross-sectional area may vary with the distance x from the wall. An annular fin is one that is circumferentially attached to a cylinder, and its cross-section varies with radius from the wall of the cylinder. The foregoing fin types have rectangular cross-sections, whose area may be expressed as a product of the fin thickness t and the width w for straight fins or the circumference 2πr for annular fins. In contrast, a pin fin or spine is an extended surface of a circular cross-section. Pin fins may also be of uniform or no uniform cross-section. In any application, selection of a particular fin configuration may depend on space, weight, manufacturing, and cost considerations, as well as on the extent to which the fins reduce the surface convection coefficient and increase the pressure drop associated with the flow over the fins [1].

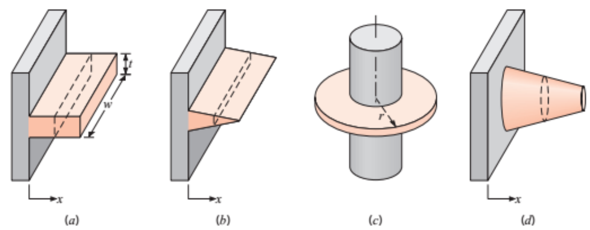

Fig-1: Fin Configurations. (a) Straight Fin of uniform Cross -Section. (b) Straight Fin of Non-uniform Cross -Section. (c) Annular Fin. (d) Pin Fin.

The objectives of the present paper is to understand the finite volume method and its application in heat transfer i.e. the insulated and uniform cross-sectional area of the fin tip, and compare the numerical results with the analytical approach.

2. ANALYTICAL MODEL

As a simple illustration, consider a pin fin having the shape of a rod whose base is attached to a wall at surface temperature *Ts* figure (2). The fin is cooled along its surface by a fluid at temperature *T*∞. The fin has a uniform cross-sectional area *A* and is made of a material having uniform conductivity *k*; the heat transfer coefficient between the surface of the fin and the fluid is hc. We will assume that transverse temperature gradients are so small that the temperature at any cross-section of the rod is uniform, that is, T = T (x) only. As shown in Gardner [2], even in a relatively thick fin the error in a one-dimensional solution is less than 1%.

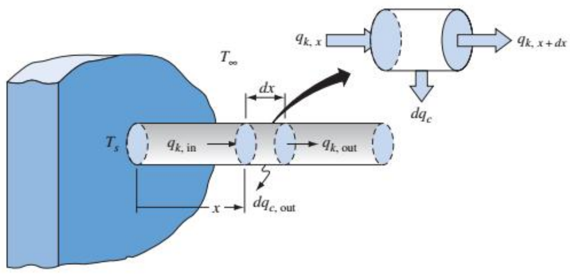

Fig-2: Schematic Diagram of Fin Protruding from a Wall.

To derive an equation for temperature distribution, we make a heat balance for a small element of the fin. Heat flows by conduction into the left face of the element, while heat flows out of the element by conduction through the right face and by convection from the surface. Under steady-state conditions,

rate of heat flow by conduction into element at x	=	rate of heat flow by conduction out of element at $x + dx$	+	rate of heat flow by convection from surface between $x + dx$

In symbolic form, this equation becomes

$$q_{k,x} = q_{k,x+dx} + dq_c$$

or

$$\left|-kA\frac{dT}{dx}\right|_x = \left|-kA\frac{dT}{dx}\right|_{x+dx} + h_c P dx[T(x) - T_\infty] \qquad (1)$$

Where *P* is the perimeter of the pin and *P dx* is the pin surface area between *x* and *dx*. If *k* and h_c are uniform, equation (1) simplifies to the form

$$\frac{d^2T(x)}{dx^2} - \frac{h_c P}{kA}[T(x) - T_\infty] = 0 \quad (2)$$

It will be convenient to define an excess temperature of the fin above the environment θ(x) = [T(x) - T∞], and transform equation (2) into the form

$$\frac{d^2\boldsymbol{\theta}}{dx^2} - m^2\boldsymbol{\theta} = 0 \quad (3)$$

Where $m^2 = \frac{h_c P}{kA}$

Equation (3) is a linear, homogeneous, second-order differential equation whose general solution is of the form

$$\theta(x) = C_1 e^{mx} + C_2 e^{-mx} \quad (4)$$

To evaluate the constants C1 and C2 it is necessary to specify appropriate boundary conditions. One condition is that at the base (x = 0) the fin temperature is equal to the wall temperature, or

$$\theta(0) = T_s - T_\infty = \theta_s$$

The other boundary condition depend on the physical condition at the end of the fin. We will treat the following four cases:

1- *The fin is very long* and the temperature *at the end approaches the fluid temperature:*

$$\theta \ at \ x \rightarrow \infty$$

2- The end of the fin is insulated:

$$\frac{dT}{dx} = 0 \ \ at \ \ x = L$$

3- The temperature at the end of the fin is fixed:

$$\theta = \theta_L \quad at \quad x = L$$

4- The tip loses heat by convection:

$$\left|-kA\frac{d\theta}{dx}\right|_{x=L} = h_{c,L}\theta_L$$

Figure (3) illustrates schematically the cases described by these conditions at the tip.

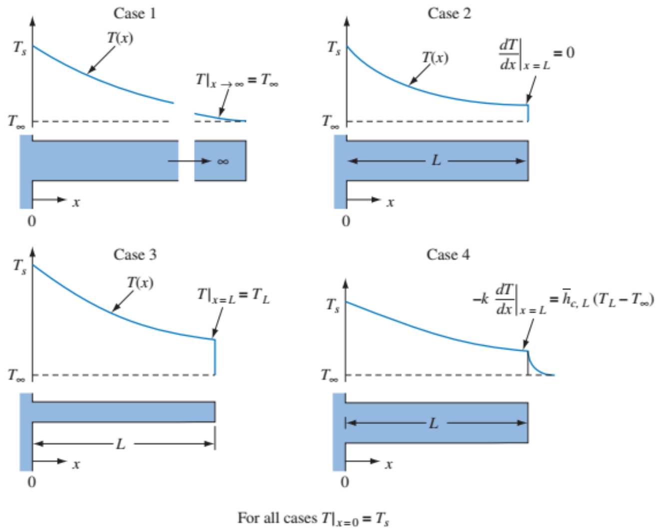

Fig-3: Schematic Representation of four Boundary Conditions at the tip of a Fin.

Usually, we are interested not only in the temperature distribution but also in the total rate of heat transfer to or from the fin. The rate of heat flow can be obtained by two different methods. Since the heat conducted across the root of the fin must equal the heat transferred by convection from the surface of the rod to the fluid [3].

$$q_{\text{fin}} = \left|-kA\frac{d\theta}{dx}\right|_{x=0} = \int_0^{\infty} h_c P[T(x) - T_\infty]dx$$
$$= \int_0^{\infty} h_c P\theta(x)dx \quad (5)$$

In this paper we discuss only the rod is of finite length but the heat loss from the end of the rod is neglected, or if the end of the rod is insulated, the second boundary condition requires that the temperature gradient at x = L be zero, that is, dT/dx = 0 at x = L. These conditions require that

$$\left(\frac{d\theta}{dx}\right)_{x=L} = 0 = mC_1e^{mL} - mC_2e^{-mL}$$

Solving this equation for condition 2 simultaneously with the relation for condition 1, which required that

$$\theta(0) = \theta_s = C_1 + C_2$$

Yields

$$C_1 = \frac{\theta_s}{1 + e^{2mL}} \qquad C_2 = \frac{\theta_s}{1 + e^{-2mL}}$$

Substituting the above relations for C1 and C2 into equation (4) gives the temperature distribution

$$\frac{\theta}{\theta_s} = \left(\frac{e^{mx}}{1 + e^{2mL}} + \frac{e^{-mx}}{1 + e^{-2mL}}\right) = \left(\frac{\cosh m(L-x)}{\cosh(mL)}\right) \quad (6)$$

3. NUMERICAL MODEL

An immense number of analytical solutions for conduction heat-transfer problems have been accumulated in the literature over the past 150 years [4]. Even so, in many practical situations, the geometry or boundary conditions are such that an analytical solution has not been obtained at all, or if the solution has been developed, it involves such a complex series solution that numerical evaluation becomes exceedingly difficult. For such situations, the most fruitful approach to the problem is one based on finite volume techniques, the basic principles of which we shall outline in this paper.

Availability of high-speed computers makes it possible for today's engineers to find answers for highly complicated problems of thermodynamics and heat transfer. Though a large number of software are available for thermal analysis, it is required to understand the calculations that are carried out by the software in the background to avoid any possible pitfall. The finite volume method is one of the methods that is used as a numerical method of finding answers to some of the classical problems of heat transfer. Present section deals with the fundamental aspects of the finite volume method and its application in the study of fins. The numerical methods for solving differential equations are based on replacing the differential equations by algebraic equations. We are going to solve the one-dimensional diffusion equation for temperature T, by the finite volume method.in these cases, we discuss the cooling of the circular fin using convective heat transfer along its length. Convection gives rise to temperature-dependent heat loss or heat sink term in the governing equation. Shown in figure (2) is a cylindrical fin with uniform cross-sectional area A. The base temperature TS and the end is insulated. The fin is exposed to an ambient temperature T∞. One-dimensional heat transfer in this situation is governed by

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) - hP(T - T_\infty) = 0 \qquad (7)$$

When *kA=constant*, the governing equation (7) can be written as

$$\frac{d}{dx}\left(kA\frac{dT}{dx}\right) - m^2(T - T_\infty) = 0 \quad where\ m^2 = \frac{hp}{kA} \qquad (8)$$

Integration of the above equation over a control volume use finite volume method for one-dimensional steady-state diffusion shown in figure (4a).

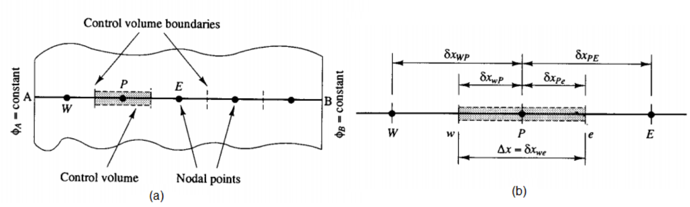

Fig-4: (a) Control Volume Figure and (b) Convention of CFD Method

The first step in the finite volume method is to divide the domain into discrete control volumes. Let us place a number of nodal points in the space between A and B. The boundaries (or faces) of control volume are positioned mid-way between adjacent nodes. Thus each node is surrounded by a control volume or cell. The usual convention of CFD methods is shown in figure (4b). A general nodal point is identified by P and its neighbors in One-dimensional geometry, the nodes to the west and east, are identified by W and E respectively. Now, integrate the equation (8) over the control volume gives

$$\int_{\Delta V} \frac{d}{dx}\left(\frac{dT}{dx}\right) dV - \int_{\Delta V} m^2 (T - T_\infty)\, dV = 0 \qquad (9)$$

The first integral of the above equation can use the linear approximation seem to be the obvious and simplest way of calculating interface values and the gradients. This practice is called central differencing [5]; the second integral due to the source term in the equation is evaluated by assuming that the integral is locally constant within each control volume.

$$\left[\left(A\frac{dT}{dx}\right)_e - \left(A\frac{dT}{dx}\right)_w\right] - [m^2(T_P - T_\infty)A\delta x] = 0$$

First we develop a formula by introducing the usual linear approximation for the temperature gradient. Subsequent division by cross-sectional are A gives

$$\left[\left(\frac{T_E - T_P}{\delta x}\right) - \left(\frac{T_P - T_W}{\delta x}\right)\right] - [m^2(T_P - T_\infty)\,\delta x] = 0$$

This can be re-arranged as

$$\left(\frac{1}{\delta x} + \frac{1}{\delta x}\right)T_P = \left(\frac{1}{\delta x}\right)T_W + \left(\frac{1}{\delta x}\right)T_E + m^2\delta x T_\infty - m^2\,\delta x\, T_P \qquad (10)$$

For interior nodal using general form

$$a_P T_P = a_W T_W + a_E T_E + S_u \qquad (11)$$

a_W	a_E	a_P	S_P	S_u
$\frac{1}{\delta x}$	$\frac{1}{\delta x}$	$a_W + a_E - S_P$	$-m^2\delta x$	$m^2\delta x T_\infty$

Next we apply the boundary condition at the left and write nodes point A and B. At node A the west control volume boundary is kept at a specified temperature (T_S).

$$\left[\left(\frac{T_E - T_P}{\delta x}\right) - \left(\frac{T_P - T_S}{\delta x/2}\right)\right] - [m^2(T_P - T_\infty)\,\delta x] = 0$$

The coefficients of the discretized equation at boundary node A are

a_W	a_E	a_P	S_P	S_u
0	$\frac{1}{\delta x}$	$a_W + a_E - S_P$	$-m^2\delta x - \frac{2}{\delta x}$	$m^2\delta x T_\infty + \frac{2}{\delta x}T_S$

At node B the flux across the east boundary is zero since the east side of the control volume is an insulated boundary:

$$\left[0 - \left(\frac{T_P - T_W}{\delta x}\right)\right] - [m^2(T_P - T_\infty)\,\delta x] = 0$$

Hence the east coefficient is zero. There are no additional source terms associated with the zero flux boundary condition. The coefficients at boundary node B are given by

a_W	a_E	a_P	S_P	S_u
$\frac{1}{\delta x}$	0	$a_W + a_E - S_P$	$-m^2\delta x$	$m^2\delta x T_\infty$

4. RESULTS AND DISCUSSIONS

One-dimensional heat transfer problems can be approached in a number of ways. The analytical approach using the direct solutions of differential equations. The analysis can get quite complex depending on boundary conditions. A more practical approach is the use of numerical methods. The ***finite volume method*** seems to provide a good approach as using this method we can model fairly complex problems with a variety of boundary conditions using MATLAB software program. With the help of ***finite volume method*** one. In this paper, we discuss the cooling of a circular fin by means of convective heat transfer along its length. Convection gives rise to a temperature-dependent heat loss or heat sink term in the governing equation [5]. Shown in figure (5) is a cylindrical fin with uniform cross-sectional area A. The base is at a temperature of 100◦C and the end is insulated. The fin is exposed to the ambient temperature of 20◦C.the lengths of the fin is 1 m, and hp/kA=25 m-2 (note kA is constant).

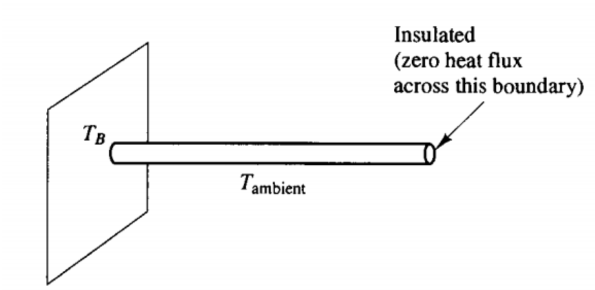

Fig-5: Geometry of the Fin.

The fin information can be applied to compare the analytical solution and the approximate solution, a circular problem fin was examine used the analytical solution, and the nodal temperature distribution of circular fin is shown in figure (6).

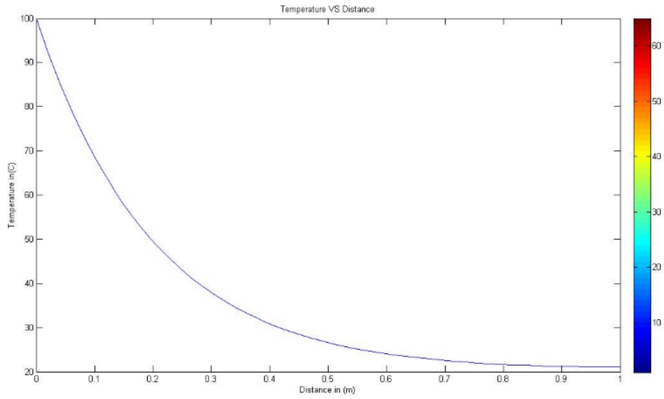

Fig-6: Temperature Distribution for Analytical Solution.

The results for the numerical determination of the one-dimensional first derivative heat transfer problems, which are derived from ***finite volume method*** at different step sizes, when using a circular problem fin shown in figure (7). First, the solution was not convergent, but after increasing the number of the elements to 50 using the MATLAB program, and applied GAUSS-SEIDEL iteration method to solve and calculate the temperatures at different locations along rod. The shape of the curve has become constant and does not change by increasing the number of elements.

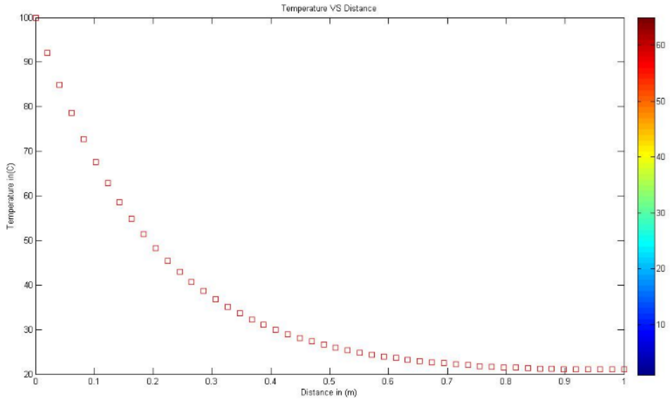

Fig-7: Temperature Distribution for Numerical Solution.

The analytical solution was drawn with the approximate solution to illustrate the difference between the two solutions as shown in figure (8), the error rate between the two solutions did not exceed 0.71%, that means, the numerical methods (***finite volume method***) seems to provide a good approach as using for complex problems of one-dimensional heat transfer problems.

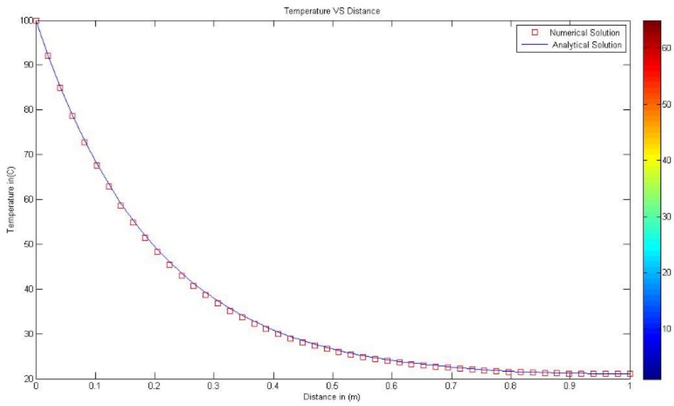

Fig-8: Analytical and Numerical Solution Comparison.

In figure (9) the residual error is plotted against the number of iteration for explained the convergence criteria values, it's found that the solution has a divergence phenomenon at the first stages with maximum value of residual of 110 and then decreases rapidly until convergence and the stability stage is reached after approximately 300 iterations at error of 10^{-7}.

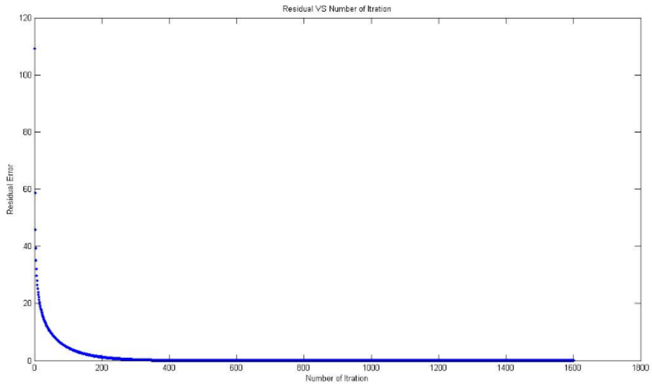

Fig-9: The Residual Error.

The heat flux losses from the fin along the different locations are shown in figure (10). The heat flux is not constant throughout the fin surface area. The heat flux is a maximum near to the wall and decreases rapidly. The heat flux far away from the base of the fin becomes almost constant.

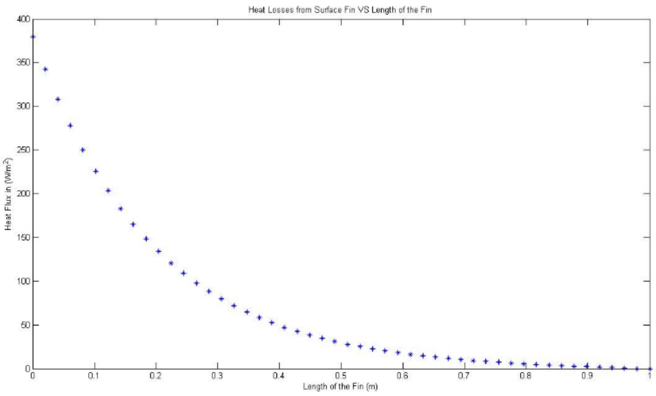

Fig-10: ***Heat Flux from the Surface of Fin.***

5. CONCLUSIONS

In this study, the numerical solution of the insulated and uniform cross-sectional area of the fin was verified by the exact solution. The one-dimensional heat transfer differential equation was solved analytically and numerically, used direct numerical solutions for analytical solution of differential equations and the ***finite volume method*** was used to model the numerical solution. A program was written in MATLAB based on the mathematical model and applied GAUSS-SEIDEL iteration solver. The temperature distributions along the fin were plotted for the analytical and the numerical models, it was found that the approximate solution is very close to the analytical solution. With the help of the ***finite volume method*** concept, any thermal system can be modeled by applying the energy balance equation on the volume element of the specified node.

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