The Pendant Domination and Complementary Pendant Domination Numbers of Graphs: It's Results

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Abstract: We investigate the pendant domination and complementary pendant domination numbers of graphs in the present study. In this paper, the pendant domination and complementary pendant domination numbers for several classes of graphs are computed.

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1. Introduction

The graphs considered here are all finite, non-trivial, undirected and connected simple graph. As usual n and m denote the number of vertices and edges of a graph G. Any undefined term or notation in this paper can be found in [1, 2]. The degree of a vertex v in a graph G denoted by $degv$ is the number of edges of G incident with v. For each vertex $v \in V$, the open neighborhood of v is the set $N(v)$ containing all the vertices u adjacent to v. A vertex of a degree one is called a pendant vertex.

The extensive study of dominating sets in graph theory began around 1960. In 1977, Cockayne and Hedetniemi extensively surveyed the results of dominating sets in graphs. The concept of domination in graphs has attracted many researchers in graph theory and many domination parameters like total, connected, paired domination were defined and studied in last few decade. The reader is referred to [4, 5, 6, 15] for survey or results on domination.

A set $S \subseteq V(G)$ is called a dominating set of G if each vertex of $V - S$ is adjacent to at least one vertex of S. The domination number of a graph G denoted as $\gamma(G)$ is the minimum cardinality of a dominating set in G [3].

The concept of a pendant domination number and complementary pendant domination number of graphs are defined as follows:

Definition 1.1 *[9] A dominating set S in G is called a pendant dominating set if $\langle S \rangle$ contains at least one pendant vertex. The minimum cardinality of a pendant dominating set is called the pendant domination number denoted by $\gamma_{pe}(G)$.*

Definition 1.2 *[10] A dominating set S in G is called a complementary pendant dominating set if $\langle V - S \rangle$ contains at least one pendant vertex. The minimum cardinality of a complementary pendant dominating set is called the complementary pendant domination number of G, denoted by $\gamma_{cpd}(G)$.*

For more details on the pendant domination number see [11, 12].

In this work, we investigate pendant domination number of splitting graph of path, cycle, star, double star and complete bipartite graph and we discuss pendant domination number of some graphs. We also identify complementary pendant domination number of some graphs.

2. Main results

We now proceed to compute $\gamma_{pe}(G)$ and $\gamma_{cpd}(G)$ for some graphs.

Definition 2.1 *[13] For a graph G the splitting graph $S'_p(G)$ of a graph G is obtained by adding a new vertex v' corresponding to each vertex v of G such that $N(v) = N(v')$, where $N(v)$ and $N(v')$ are the neighborhood sets of v and v', respectively.*

Theorem 2.1 *For any path P_n of order $n \geq 3$, the following cases are obtained*

$$\gamma_{pe}(S_p(P_n)) = \begin{cases} \frac{n}{2}, & if \quad n \equiv 0(mod \quad 4) \ ; \\ \lceil\frac{n}{2}\rceil + 1, & if \quad n \equiv 1(mod \quad 4) \ ; \\ \frac{n}{2} + 1, & if \quad n \equiv 2(mod \quad 4) \ ; \\ \lceil\frac{n}{2}\rceil, & if \quad n \equiv 3(mod \quad 4). \end{cases}$$

Proof. Let $\{v_1, v_2, v_3, \ldots, v_n\}$ be the vertex set of a path P_n and $\{u_1, u_2, u_3, \ldots, u_n\}$ be the new vertices corresponding to $\{v_1, v_2, v_3, \ldots, v_n\}$ which are added to obtain $S_p(P_n)$. The graph $S_p(P_8)$ is shown in Figure 1 for better understanding of the notation and arrangement of vertices. We consider the following cases:

Case 1: If $n \equiv 0\ (mod\ 4)$, then $n = 4k$ and $k \geq 1$.

Consider $S = \{v_{2+4i}, v_{3+4i}, 0 \leq i \leq k-1\}$ is a pendant dominating set of $S_p(P_n)$ such that $|S| = \frac{n}{2}$.

We arrange the set S to a minimal pendant dominating set, each v_{2+4i} is adjacent with v_{1+4i}, v_{3+4i}, u_{1+4i} and u_{3+4i}. If v_{2+4i} is removed from set S, then the following vertices v_{1+4i} and v_{3+4i} also, u_{1+4i} and u_{3+4i} are not dominated by any vertex. Thus S is a minimal pendant dominating set with minimum cardinality. Then, $\gamma_{pe}(S_p(P_n)) = \frac{n}{2}$.

Case 2: If $n \equiv 1\ (mod\ 4)$, then $n = 4k + 1$ and $k \geq 1$.

Assume that $S = \{v_{2+4i}, v_{3+4i}, 0 \leq i \leq k-1\} \cup \{v_n, u_n\}$ is a pendant dominating set of $S_p(P_n)$ such that $|S| = \lceil\frac{n}{2}\rceil + 1$. It is clear that S is minimal pendant dominating set. Then, $\gamma_{pe}(S_p(P_n)) = \lceil\frac{n}{2}\rceil + 1$.

Case 3: If $n \equiv 2\ (mod\ 4)$, then $n = 4k + 2$ and $k \geq 1$.

Consider $S = \{v_{2+4i}, v_{3+4i}, 0 \leq i \leq k-1\} \cup \{v_{n-1}, v_n\}$, is a pendant dominating set of $S_p(P_n)$ such that $|S| = \frac{n}{2} + 1$. We claim that set S is a minimal pendant dominating set. One can observe that each v_{1+4i}, v_{3+4i}, u_{1+4i} and u_{3+4i} are adjacent to v_{2+4i} and by removal of v_{2+4i} from set S. The following vertices say V_{1+4i} and V_{3+4i} also u_{1+4i} and u_{3+4i} are not dominated by any vertex of S. Therefore, $\gamma_{pe}(S_p(P_n)) = \frac{n}{2} + 1$.

Case 4: If $n \equiv 3\ (mod\ 4)$, then $n = 4k + 3$, and $k \geq 1$.

Assume that $S = \{v_{2+4i}, v_{3+4i}, 0 \leq i \leq k\}$ is a pendant dominating set of $S_p(P_n)$ such that $|S| = \lceil\frac{n}{2}\rceil$. We will show that S is minimum. Since $N(v_{2+4i}) = \{v_{1+4i}, v_{3+4i}, u_{1+4i}, u_{3+4i}\}$ and $N(v_{3+4i}) = \{v_{2+4i}, v_{4+4i}, u_{2+4i}, u_{4+4i}\}$ and removal of v_{2+4i} the vertices v_{1+4i} and u_{1+4i} are not dominated by any vertex of S. Hence, S is a minimal pendant dominating set with minimum cardinality. Then, $\gamma_{pe}(S_p(P_n)) = \lceil\frac{n}{2}\rceil$.

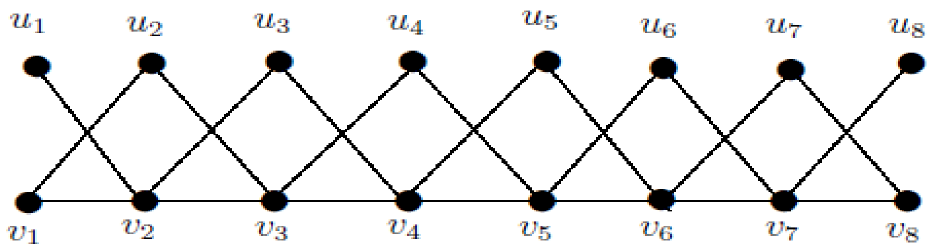

Figure 1: $S_p(P_8)$

Remark 2.1 *For any path* P_n *of order* $n \geq 3$ *then,* $\gamma_{pe}(S_p(P_n)) = \gamma_{cpd}(S_p(P_n))$.

Theorem 2.2 *For any cycle* C_n *of order* $n \geq 3$ *then,*

$$\gamma_{pe}(S_p(C_n)) = \begin{cases} \frac{n}{2}, \ if \ \ n \equiv 0(mod \ \ 4) \ ; \\ \left\lceil \frac{n}{2} \right\rceil + 1, \ if \ \ n \equiv 1(mod \ \ 4) \ ; \\ \frac{n}{2} + 1, \ if \ \ n \equiv 2(mod \ \ 4) \ ; \\ \lceil \frac{n}{2} \rceil, \ if \ \ n \equiv 3(mod \ \ 4). \end{cases}$$

Proof. Let $v_1, v_2, v_3, \ldots, v_n$ be the vertices of cycle C_n and $u_1, u_2, u_3, \ldots, u_n$ be the vertices corresponding to $v_1, v_2, v_3, \ldots, v_n$ that are been added to obtain $S_p(C_n)$. The graph $S_p(C_8)$ is shown in Figure 2 for better understanding of the notation and arrangement of vertices. To prove the result, the following four cases are considered:

- If $n \equiv 0 \ \ (mod \ \ 4)$, then $n = 4k$ and $k \geq 1$.
  $S = \{v_{2+4i}, v_{3+4i}, 0 \leq i \leq k-1\}, |S| = \frac{n}{2}$.
- If $n \equiv 1 \ \ (mod \ \ 4)$, then $n = 4k+1$ and $k \geq 1$.
  $S = \{v_{2+4i}, v_{3+4i}, 0 \leq i \leq k-1\} \cup \{v_n, u_n\}, |S| = \lceil \frac{n}{2} \rceil + 1$.
- If $n \equiv 2 \ \ (mod \ \ 4)$, then $n = 4k+2$ and $k \geq 1$.
  $S = \{v_{2+4i}, v_{3+4i}, 0 \leq i \leq k-1\} \cup \{v_{n-1}, v_n\}, |S| = \frac{n}{2} + 1$.
- If $n \equiv 3 \ \ (mod \ \ 4)$, then $n = 4k+3$ and $k \geq 1$.
  $S = \{v_{2+4i}, v_{3+4i}, 0 \leq i \leq k\}, |S| = \lceil \frac{n}{2} \rceil$.

We claim that each S is a minimal pendant dominating set of $S_p(C_n)$. Because each v_{2+4i} is adjacent to v_{1+4i}, v_{3+4i}, u_{1+4i} and u_{3+4i}. If v_{2+4i} is removed from set S, then u_{1+4i} and u_{3+4i} arel not dominated by any vertex of S. Then S is a minimal pendant dominating set with minimum cardinality. Therefore, we get the result.

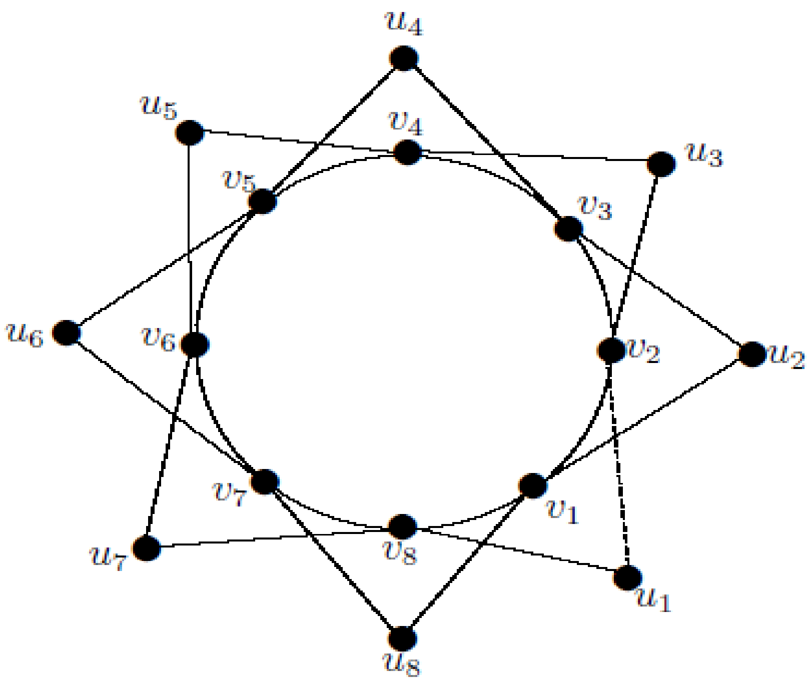

Figure 2: $S_p(C_8)$

Theorem 2.3 *For a star graph* $S_{1,n-1}$ *we get,* $\gamma_{pe}(S_p(S_{1,n-1})) = 2.$

Proof. Let $v, v_1, v_2, \ldots, v_{n-1}$ be the vertices of star $S_{1,n-1}$ and $u, u_1, u_2, \ldots, u_{n-1}$ be the vertices corresponding to $v, v_1, v_2, \ldots, v_{n-1}$ are added to obtain $S_p(S_{1,n-1})$. Suppose $S = \{v\} \cup \{v_i\}$, $1 \le i \le n-1$ is a pendant dominating set such that $|S| = 2$ and consider S is a minimum pendant dominating set. To show that S is minimal pendant dominating set, we note that $N(v) = \{v_1, v_2, \ldots v_{n-1}, u_1, u_2, \ldots, u_{n-1}\}$ and $N(v_i) = \{u, v\}$, and by removal of v, no vertex in S will dominate the following vertices $\{v_1, v_2, \ldots v_{n-1}, u_1, u_2, \ldots, u_{n-1}\}$. Also, by removal of $\{v_i\}$, $1 \le i \le n-1$ of S, the vertex $\{u\}$ will not be dominated. So, S is a minimum pendant dominating set of $S_p(S_{1,n-1})$. Then $\gamma_{pe}(S_p(S_{1,n-1})) = 2$.

Theorem 2.4 *For a double star* $S_{n,m}$ *we obtain,* $\gamma_{pe}(S_p(S_{n,m})) = 2.$

Proof. Let $v, v_1, v_2, \ldots, v_n, u, u_1, u_2, \ldots, u_m$ be the vertices of double star $S_{n,m}$ and $v', v'_1, v'_2, \ldots, v'_n, u', u'_1, u'_2, \ldots, u'_m$ be the vertices corresponding to $v, v_1, v_2, \ldots, v_n, u, u_1, u_2, \ldots, u_m$ which are added to obtain $S_p(S_{n,m})$. Suppose $S = \{u, v\}$ is a pendant dominating set, then consider S as a minimum pendant dominating set. Since $N(v) = \{u', v_1, v_2, \ldots v_n, v'_1, v'_2, \ldots, v'_n\}$ and $N(u) = \{v', u_1, u_2, \ldots u_m, u'_1, u'_2, \ldots, u'_m\}$.

If v is removed of S, the vertices $\{v_1, v_2, \ldots v_n, v'_1, v'_2, \ldots, v'_n\}$ and $\{u'\}$ will not be dominated by any vertex of S. Also If u is removed of S, the vertices $\{u_1, u_2, \ldots u_m, u'_1, u'_2, \ldots, u'_m\}$ and $\{v'\}$ will not be dominated by any vertex of S. So, S is a minimum pendant dominating set of $S_p(S_{n,m})$. Then $\gamma_{pe}(S_p(S_{n,m})) = 2$.

Theorem 2.5 *For a bipartite graph* $K_{n,m}$ *we obtain,* $\gamma_{pe}(S_p(K_{n,m})) = 2.$

Proof. Let $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m$ be the vertices of bipartite graph $K_{n,m}$ and $u'_1, u'_2, \ldots, u'_n$, $v'_1, v'_2, \ldots, v'_m$ be the vertices corresponding to $u_1, u_2, \ldots, u_n, v_1, v_2, \ldots, v_m$ are added to obtain $S_p(K_{n,m})$. Consider S is a pendant dominating set of any two vertices of $K_{n,m}$ such that the first vertex of the partite V and the second vertex from the partite U, this means that we can choose any two vertices as follows: $S = \{u_1, v_1\}$ and $|S| = 2$. Now we consider S is a minimum pendant dominating set. Since $N(u_1) = \{v_1, v_2, \dots v_m, v'_1, v'_2, \dots, v'_m\}$ and $N(v_1) = \{u_1, u_2, \dots u_n, u'_1, u'_2, \dots, u'_n\}$.

If u_1 is removed of S, the vertices $\{v_1, v_2, \dots v_m, v'_1, v'_2, \dots, v'_m\}$ will not be dominated by any vertex of S. Also If v_1 is removed of S, the vertices $\{u_1, u_2, \dots u_n, u'_1, u'_2, \dots, u'_n\}$ will not be dominated by any vertex of S. So, S is a minimum pendant dominating set of $S_p(K_{n,m})$. Then $\gamma_{pe}(S_p(K_{n,m})) = 2$.

Definition 2.2 *[2] The Cartesian product* $G \times H$ *of graphs* G *and* H *has* $V(G) \times V(H)$ *as its vertex set and* (u_1, u_2) *is adjacent to* (v_1, v_2) *if either* $u_1 = v_1$ *and* u_2 *is adjacent to* v_2 *or* $u_2 = v_2$ *and* u_1 *is adjacent to* v_1 *.*

Theorem 2.6

$$\gamma_{pe}(K_2 \times P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1, if \quad n \equiv 0 (mod \quad 3) \; ; \\ \lceil \frac{2n-1}{3} \rceil, \; if \quad n \equiv 1 (mod \quad 3) \; ; \\ \lceil \frac{n}{2} \rceil + 2, \; if \quad n \equiv 2 (mod \quad 3). \end{cases}$$

Proof. Let P_n be a path with vertices $v_1, v_2, \dots, v_n$ and complete graph K_2 with vertices u_1, u_2, we denote vertices (u_1, v_i) by w_{1i}, $1 \le i \le n$ and (u_2, v_i) by w_{2i}, $1 \le i \le n$. The graph $K_2 \times P_n$ is shown in Figure 3. The following cases are considered:

- If $n \equiv 0 \quad (mod \quad 3), n = 3k$ and $k \ge 1$,
  $S = \{w_{1(2+3i)}, w_{2(2+3i)}, 0 \le i < k\}, |S| = \lceil \frac{n}{2} \rceil + 1.$

- If $n \equiv 1 \quad (mod \quad 3), n = 3k + 1$ and $k \ge 1$,
  $S = \{w_{1(2+3i)}, w_{2(2+3i)}, 0 \le i < k\} \cup \{w_{1n}\}, |S| = \lceil \frac{2n-1}{3} \rceil.$

- If $n \equiv 2 \quad (mod \quad 3), n = 3k + 2$ and $k \ge 1$,
  $S = \{w_{1(2+3i)}, w_{2(2+3i)}, 0 \le i < k\}, |S| = \lceil \frac{n}{2} \rceil + 2.$

In all the above cases, S is a pendant dominating set of $K_2 \times P_n$. If any vertex of S is removed, there exist some vertex that will not be dominated by any vertex of S. So, S is a minimum pendant dominating set and this completes the proof.

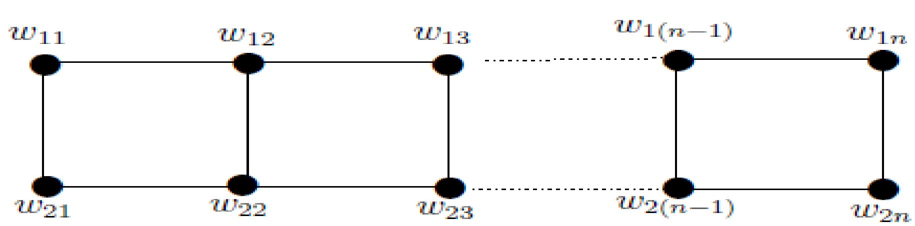

Figure 3: $K_2 \times P_n$

Theorem 2.7

$$\gamma_{pe}(K_2 \times C_n) = \begin{cases} \frac{2n}{3}, & if \quad n \equiv 0 (mod \quad 3) \; ; \\ \lceil \frac{2n-1}{3} \rceil, & if \quad n \equiv 1 (mod \quad 3) \; ; \\ \frac{2n+2}{3}, & if \quad n \equiv 2 (mod \quad 3). \end{cases}$$

Proof. Let C_n be a cycle with vertices $v_1, v_2, \dots, v_n$ and complete graph K_2 with vertices u_1, u_2. We denote vertices (u_1, v_i) by w_{1i}, $1 \le i \le n$ and (u_2, v_i) by w_{2i}, $1 \le i \le n$. The graph $K_2 \times C_n$ is shown in Figure 4. We consider a subset S of $V(K_2 \times C_n)$ as below:

- If $n \equiv 0 \quad (mod \quad 3)$, then $n = 3k$ and $k \ge 1$.
  $$S = \{w_{1(2+3i)}, w_{2(2+3i)}, 0 \le i < k\}, |S| = \frac{2n}{3}.$$
- If $n \equiv 1 \quad (mod \quad 3)$, then $n = 3k + 1$ and $k \ge 1$.
  $$S = \{w_{1(2+3i)}, w_{2(2+3i)}, 0 \le i < k\} \cup \{w_{1n}\}, |S| = \lceil \frac{2n-1}{3} \rceil.$$
- If $n \equiv 2 \quad (mod \quad 3)$, then $n = 3k + 2$ and $k \ge 1$.
  $$S = \{w_{1(2+3i)}, w_{2(2+3i)}, 0 \le i \le k\}, |S| = \frac{2n+2}{3}.$$

The proof of minimality of S is similar to that of the Theorem 2.6.

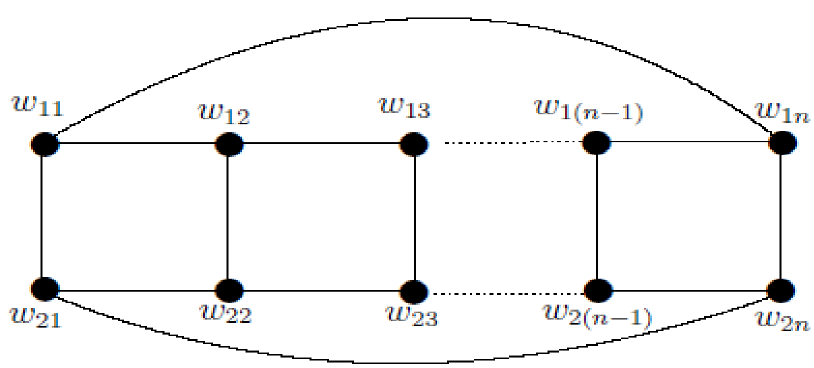

Figure 4: $K_2 \times C_n$

Definition 2.3 *[2] The line graph $L(G)$ of G has the edges of G as its vertices which are adjacent in $L(G)$ if and only if the corresponding edges are adjacent in G.*

Definition 2.4 *[8] The middle graph $M(G)$ of a graph G is the graph whose set of vertices is the union of the set of vertices and edges of G in which two vertices are adjacent if they are adjacent edges of G or one is a vertex of G and other is an edge of G incident with it.*

Theorem 2.8 *If $M(P_n)$ is a middle graph of path P_n, then*

$$\gamma_{pe}(M(P_n)) = \begin{cases} \frac{n}{2}+1, & if \;\; n \;\; is \;\; even \;\; ; \\ \lceil\frac{n}{2}\rceil, & if \;\; n \;\; is \;\; odd. \end{cases}$$

Proof. Let $V(M(P_n)) = \{w_1, w_2, \ldots, w_{n-1}\} \cup \{v_1, v_2, \ldots, v_n\}$. We consider two cases:

Case 1: If n is even, we consider $S = \{w_1, w_2, w_4, w_6, w_8, \ldots, w_{n-2}, w_{n-1}\}$ a pendant dominating set of $M(P_n)$ such that $|S| = \frac{n}{2}+1$. Let S be a minimum pendant dominating set. If w_1 or w_{n-1} are removed from set S, then none of the vertices in set S will dominate the vertices v_1 and v_n. Hence, S is minimum pendant dominating set. Therefore,$\gamma_{pe}(M(P_n)) = \frac{n}{2}+1$.

Case 2: If n is odd, consider $= \{w_1, w_2, w_4, w_6, w_8, \ldots, w_{n-3}, w_{n-1}\}$ a pendant dominating set of $M(P_n)$ such that $|S| = \lceil\frac{n}{2}\rceil$. To show that S is minimum pendant dominating set. If w_2 or w_{n-3} are removed from set S, then none of the vertices in set S will dominate vertices v_3,v_{n-2} and v_{n-3}. So, $\gamma_{pe}(M(P_n)) = \lceil\frac{n}{2}\rceil$.

Remark 2.2 *If $M(P_n)$ is a middle graph of path P_n then, $\gamma_{pe}(M(P_n)) = \gamma_{cpd}(M(P_n))$.*

Theorem 2.9 *If $\gamma_{pe}(M(C_n)$ is a pendant domination number of middle graph of cycle C_n, then we obtain the following results,*

$$\gamma_{pe}(M(C_n)) = \begin{cases} \lceil\frac{n}{2}\rceil + 1, & if \;\; n \equiv 0(mod \;\; 3) \;\; ; \\ \lceil\frac{2n-1}{3}\rceil, & if \;\; n \equiv 1(mod \;\; 3) \;\; ; \\ \lceil\frac{2n-2}{3}\rceil, & if \;\; n \equiv 2(mod \;\; 3). \end{cases}$$

Proof. Let $V(M(C_n)) = \{w_1, w_2, \ldots, w_n\} \cup \{v_1, v_2, \ldots, v_n\}$. We have the following cases:

- If $n \equiv 0 \;\; (mod \;\; 3)$, then $n = 3k$ and $k \geq 1$,
$S = \{w_{1+3i}, w_{2+3i}, 0 \leq i < k\}$, $|S| = \lceil\frac{n}{2}\rceil + 1$.
- If $n \equiv 1 \;\; (mod \;\; 3)$, then $n = 3k+1$ and $k \geq 1$,
$S = \{w_{1+3i}, w_{2+3i}, 0 \leq i < k\} \cup \{v_n\}$, $|S| = \lceil\frac{2n-1}{3}\rceil$.
- If $n \equiv 2 \;\; (mod \;\; 3)$, then $n = 3k+2$ and $k \geq 1$,
$S = \{w_{1+3i}, w_{2+3i}, 0 \leq i < k\} \cup \{w_{n-1}\}$, $|S| = \lceil\frac{2n-2}{3}\rceil$.

In all the above cases, S is a pendant dominating set of $M(C_n)$. To show that S is minimum, let w_{1+3i} or w_{2+3i} be removed from set S, such that no vertex in set S dominates vertices v_{2+3i}. Hence, S is a minimum pendant dominating set of $M(C_n)$. Hence, we get the result.

Definition 2.5 *[7] A firefly graph* $F_{s,t,n-2s-2t-1}(s \geq 0, t \geq 0 \text{ and } n-2s-2t-1 \geq 0)$ *is a graph of order* n *that consists of* s *triangles,* t *pendant paths of length 2 and* $n-2s-2t-1$ *pendant edges sharing a common vertex.*

Theorem 2.10 *Let* $\gamma_{pe}(F_{s,t,n-2s-2t-1})$ *be a pendant domination number for a firefly graph. Then the following cases satisfies,*

$$\gamma_{pe}(F_{s,t,n-2s-2t-1}) = \begin{cases} 2, & if \quad s=0, \quad t=0, \quad n-1>0; \\ t+1, & if \quad s=0, \quad t>0, \quad n-2t-1>0; \\ 2, & if \quad s>0, \quad t=0, \quad n-2s-1>0; \\ t+1, & if \quad s>0, \quad t>0, \quad n-2s-2t-1>0. \end{cases}$$

Proof. Let ζ_n be the set of all firefly graphs $F_{s,t,n-2s-2t-1}(s \geq 0, t \geq 0 \text{ and } n-2s-2t-1 \geq 0)$ which is shown in Figure 5 below. Let u be the common vertex of $F_{s,t,n-2s-2t-1}$. We have the following cases:

Case 1: If $s=0, t=0$, then $F_{0,0,n-1} \cong S_{1,n-1}$. Therefore, $\gamma_{pe}(F_{0,0,n-1}) = 2$.

Case 2: If $s=0, t>0$, then consider $S = \{u, r_1, r_{22} r_{33}, \ldots, r_{tt}\}$, and $|S| = t+1$. The vertices $\{n_1, n_2, \ldots, n_{n-2s-2t-1}\}$ are dominated by the vertex u and if we remove r_1 from S, the vertex r_{11} is not dominated. So, S is a minimum pendant dominating set.

Case 3: If $s>0, t=0, n-2s-1>0$, then assume $S = \{u, z_{11}\}$. It is clear that S is a minimum pendant dominating set. Therefore, $|S| = 2$.

Case 4: If $s>0, t>0, n-2s-2t-1>0$. In this case, the proof is similar to case 2. Hence, $|S| = t+1$.

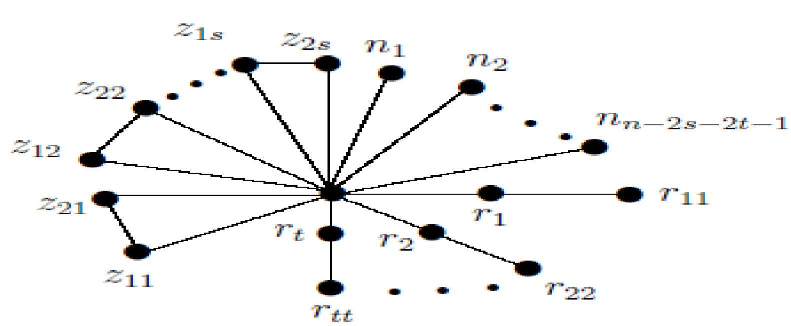

Figure 5: $F_{s,t,n-2s-2t-1}$

Theorem 2.11 *If* $\gamma_{cpd}(F_{s,t,n-2s-2t-1})$ *is a complement pendant domination number of firefly graph, then*

$$\gamma_{cpd}(F_{s,t,n-2s-2t-1}) = t+1.$$

Proof. Consider $S = \{u, r_1, r_2, \dots, r_t\}$ is complementary pendant dominating set of $F_{s,t,n-2s-2t-1}$ such that $|S| = t + 1$ and $\langle V - S \rangle = sK_2 \cup tK_1 \cup (n - 2s - 2t - 1)K_1$. Suppose the vertex $\{u\}$ is removed from set S, then none of the vertices in S will dominate the following vertices $n_1, n_2, \dots, n_{n-2s-2t-1}, z_{11}, z_{21}, z_{12}, z_{22}, \dots, z_{1s}, z_{2s}$. This shows that, set S is a minimum complementary pendant dominating set of $F_{s,t,n-2s-2t-1}$. Hence the proof.

Definition 2.6 *[14] A broom graph $B_{n,d}$ consists of a path n_d with d vertices, together with $n - d$ pendant vertices all adjacent to the same end vertex of n_d*

Theorem 2.12 *For a broom graph $B_{n,d}$, the following cases satisfies,*

$$\gamma_{cpd}(B_{n,d}) = \begin{cases} \frac{d}{3} + 1, & if \ \ d \equiv 0(mod \ \ 3); \\ \frac{d-1}{3} + 1, & if \ \ d \equiv 1(mod \ \ 3); \\ \frac{d-2}{3} + 1, & if \ \ d \equiv 2(mod \ \ 3). \end{cases}$$

Proof. Let $V(B_{n,d}) = \{u_1, u_2, \dots, u_d, w_1, w_2, \dots, w_{n-d}\}$ such that $u_1, u_2, \dots, u_d$ is a path on d vertices and $w_1, w_2, \dots, w_{n-d}$ are pendant vertices that are adjacent to u_d. We consider the following cases:

Case 1: If $d \equiv 0\ (mod\ 3), d = 3k$, and $k \geq 2$. Consider $S = \{u_{2+3i}, 0 \leq i < k\} \cup \{u_d\}$, is a complementary pendant dominating set and $|S| = \frac{d}{3} + 1$.

Case 2: If $d \equiv 1\ (mod\ 3), d = 3k + 1$ and $k \geq 1$. Consider $S = \{u_{2+3i}, 0 \leq i < k\} \cup \{u_d\}$ is a complementary pendant dominating set and $|S| = \frac{d-1}{3} + 1$.

Case 3: If $d \equiv 2\ (mod\ 3), d = 3k + 2$, and $k \geq 1$. Consider $S = \{u_{2+3i}, 0 \leq i < k\} \cup \{u_d\}$ is a complementary pendant dominating set and $|S| = \frac{d-2}{3} + 1$.

In all the above cases, it is clear that S is a minimum complementary pendant dominating set, removal of any vertex from S in all cases leads to non domination of some vertex of $B_{n,d}$. Hence the proof.

Theorem 2.13 *Let $K_{1,n-1}$ be a star with $n \geq 3$, and let G_r be a spider graph which is constructed by subdividing each edge once in $K_{1,n-1}$ as in Figure 6. Then, $\gamma_{cpd}(G_r) = n - 1$.*

Proof. Let G_r be a spider graph with $|V(G_r)| = 2n - 1$ and $|E(G_r)| = 2n - 2$. Consider $S = \{v_1, u_2, u_3, \dots, u_{n-1}\}$ a complementary pendant dominating set of G_r such that $|S| = n - 1$ then, $\langle G_r - S \rangle = K_2 \cup (n - 2)K_1$. Hence S is a minimum complementary pendant dominating set. Therefore, $\gamma_{cpd}(G_r) = n - 1$.

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