APPLICATION OF DIFFERENTIAL TRANSFORM METHOD FOR SOLVING DIFFERENTIAL EQUATIONS IN R-L-C CIRCUITS

MUNEER BASHA. H

Professor, Department of Mathematics, Ghousia College of Engineering, Visvesvaraya Technological University, Karnataka, India

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Abstract - *In this paper, the differential transform method is applied for solving the linear differential equations arising in the field of electrical circuit analysis. The solution is considered as an infinite series expansion, it converges rapidly to the exact solution. The method is useful for both linear and nonlinear equations. The solution of the problem can be expanded in Taylor's series, then the method determines the coefficients of Taylor's series. In illustrative examples the method is applied to R-L-C circuits*

Key*Words*: Differential transform method, Electrical circuits, linear differential equations, Taylor's series.

1. INTRODUCTION

Differential transform method is a powerful mathematical technique applied in various areas of engineering and science [2,3]. The concept of differential transform method was first introduced by Zhou [1], and it was applied for solving linear and nonlinear initial value problems in electrical circuits. Differential transform methods have key role to play in modern approach to the researchers of engineering system [3, 4, 5]. Suayip [6] have discussed on application of differential transform method to the system of integro – differential equations. Vedat Suat etal [7] have studied Lane – Emden equations by introducing singular initial value problems. And Wazwaz [8] has given a general study to obtain exact and series solutions of Lane – Emden equations. The Laplace transform method [9, 10] is especially useful in solving problems with nonhomogeneous terms of a discontinuous or impulsive in nature. The differential transform method is to find the coefficients of the Taylor's series of the function by solving the induced recursive equation from the differential equation of the system. Using differential transform method it is possible to obtain exact solution of various initial value problem occur in science and engineering fields. The literature of the theory and application of differential transform method is vast [10].

In this paper we extend the application of differential transform method to construct an analytical approximate solution of the linear differential equations of electrical circuits.
This paper is organized as follows: In section 2, the differential transform method is described. In section 3, the method is implemented to three examples, and conclusion is given in section 4.

2. DIFFERENTIAL TRANSFORMATION METHOD

The differential transformation of the k^{th} derivatives of function y(x) is defined as follows:

$$Y(k)=\frac{1}{k!}\left[\frac{d^k y(x)}{dx^k}\right]x=x_0 \qquad (2.1)$$

and y(x) is the differential inverse transformation of Y(k) defined as follows:

$$y(x)=\sum_{k=0}^{\infty} Y(k)(x-x_0)^k \qquad (2.2)$$

For finite series of terms equation (2.2) can be written as

$$y(x)=\sum_{k=0}^{N} Y(k)(x-x_0)^k \qquad (2.3)$$

The following theorems that can be deduced from equations (2.1) and (2.3):

Theorem1. If y(x) = g(x) $\pm$ h(x) then Y (k) = G (k) $\pm$ H (k).

Theorem2. If y(x) = αg(x) then Y (k) = αG (k).

Theorem3. If y(x) = $\frac{d\,g(x)}{dx}$ then Y (k) = (k+1) G (k+1).

Theorem4. If y(x) = $\frac{d^m g(x)}{dx^m}$ then Y (k) = $\frac{(k+1)!}{k!}$ G (k+m).

Theorem5. If y(x) = g(x).h(x) then Y (k)= $\sum_{l=0}^{k} G(l)H(k-l)$.

Theorem6. If y(x) = x^m then Y (k) = $\delta(k-m)$ = $\begin{cases}1 \; if\; k=m\\ 0 \; if \; k\neq m\end{cases}$

Theorem7. If y(x) = $e^{\alpha x}$ then Y (k) = $\frac{\alpha^k}{k!}$.

Theorem8. If y(x) = Sin $(\alpha x+\beta)$ then Y (k) = $\frac{\alpha^k}{k!}$ sin (k$\frac{\pi}{2}$ +β).

Theorem9. If y(x) = Cos $(\alpha x+\beta)$ then Y (k) = $\frac{\alpha^k}{k!}$ cos (k$\frac{\pi}{2}$ +β).

3. Application of Differential transformation method

In this section it is attempted to show that how the differential transformation method is useful to find the coefficients of the Taylor's series of the function by solving the induced recursive equation from the differential equation of the system. Here we have considered three examples of electrical circuits to demonstrate the differential transform method.

Example1

Consider an electrical circuit, containing a capacitance C and in inductance L. Let x be the electrical charge on the plate of the condenser C and i(t) be the current in the circuit at any time t. Then the potential drops across C and L being $\frac{x}{C}$ and $L\frac{di}{dt}$ respectively and since there is no applied e.m.f in the circuit we have by Kirchhoff's law

$$L\frac{di}{dt}+\frac{x}{C}=0 \quad (3.1)$$

As $i=\frac{dx}{dt}$, $L\frac{d^2x}{dt^2}+\frac{x}{C}=0$

$$\frac{d^2x}{dt^2}+\frac{x}{LC}=0 \quad (3.2)$$

writing $w^2 = 1/LC$ then we have the equation (3.2) as

$$\frac{d^2x}{dt^2}+w^2x=0 \quad (3.3)$$

with the initial conditions $x(0) = x_0 = 0$ and

$x'(0) = w$ (3.4)

On applying the differential transformation on both sides of equation (3.3) and using Theorems 1-9 the following recurrence relation is obtained:

$$\frac{(k+2)!}{k!}X(k+2)+w^2X(k)=0 \quad (3.5)$$

By using equations (2.1) and (3.4) the following transformed conditions at $x_0 = 0$ can be obtained:

$X(0) = 0, X(1) = w$ (3.6)

We can rewrite equation (3.5) as

$$X(k+2)=-\frac{w^2}{(k+1)(k+2)}X(k) \quad (3.7)$$

Following the recursive procedure we find

$X(k+2) = 0$ for $k = 0, 2, 4\ldots$

and $X(3)=-\frac{w^3}{6}$ (3.8)

$X(5)=\frac{w^5}{120}$ (3.9)

Using equations (3.6) to (3.9) and the inverse differential transformation eqn. (2.3) we get the solution of eqn. (3.3) as

$$x(t)=\sum_{k=0}^{\infty}X(k)t^k$$

$$=wt-\frac{(wt)^3}{3!}+\frac{(wt)^5}{5!}-\cdots$$

$$=Sinwt. \quad (3.10)$$

We here note that eqn. (3.10) is the exact solution of the eqn. (3.3).

Example 2

Consider the discharge of a condenser C through an inductance L and the resistance R. The potential drops across C, L, R are $\frac{x}{C}$, $L\frac{di}{dt}$ and Ri. Then by Kirchhoff's law we have

$$L\frac{d^2x}{dt^2}+R\frac{dx}{dt}+\frac{x}{C}=0 \quad (3.11)$$

with the initial conditions $x(0) = x_0 = 0$ and

$x'(0) = 1$ (3.12)

By rearranging the eqn. (3.11) we have

$$\frac{d^2x}{dt^2}+\alpha\frac{dx}{dt}+w^2x=0, \quad (3.13)$$

where $\alpha=\frac{R}{L}$, $w^2=\frac{1}{LC}$ such that $\alpha^2=4w^2$

(3.14)

On applying the differential transformation on both sides of eqn. (3.13) and using Theorems 1-9 the following recurrence relation is obtained:

$$\frac{(k+2)!}{k!}X(k+2)+\alpha\frac{(k+1)!}{k!}X(k+1)+w^2X(k)=0 \quad (3.15)$$

By using equations (2.1) and (3.12) the following transformed conditions at $x_0 = 0$ can be obtained:

$X(0) = 0, X(1) = 1$ (3.16)

We can rewrite equation (3.15) as

$$X(k+2)=-\frac{1}{(k+1)(k+2)}[\alpha(k+1)X(k+1)+w^2X(k)]$$

(3.17)

For k = 0, 1, 2 and 3 we get:

$$X(2)=-\frac{\alpha}{2}$$

$$X(3)=\frac{1}{6}[\alpha^2-w^2]$$

$$X(4)=-\frac{\alpha^3}{24}+\frac{\alpha w^2}{12}$$

$$X(5) = \frac{1}{120}[\alpha^4 - 3\alpha^2 w^2 + w^4]$$

By the inverse differential transformation eqn. (2.3) we have

$x(t) = \sum_{k=0}^{N} X(k)\,(x - x_0)^k$

$= X(0) + X(1)t + X(2)t^2 + X(3)t^3 + X(4)t^4 + X(5)t^5$

$= 0 + t + (-\frac{\alpha}{2})t^2 + \frac{1}{6}[\alpha^2 - w^2]t^3 + [-\frac{\alpha^3}{24} + \frac{\alpha w^2}{12}]t^4$

$+\frac{1}{120}[\alpha^4 - 3\alpha^2 w^2 + w^4]t^5$ (3.18)

Using the relation in (3.14), $\alpha^2 = 4w^2$ we obtain that

$x(t) = t[1 - \frac{wt}{1!} + \frac{(wt)^2}{2!} - \frac{(wt)^3}{3!} + \frac{(wt)^4}{4!} - \cdots]$ (3.19)

$x(t) = te^{-wt}$ (3.20)

which is the exact solution of the eqn. (3.13).

Example 3

Consider $\frac{d^2 i}{dt^2} + 2n\frac{di}{dt} + n^2 i = n^2 \sin nt$,

$0 < t < T$ (3.21)

where i is the current in a critically damped electrical circuit, $n^2 \sin nt$ is the e.m.f applied for a time T and $i(0)=0, \frac{di(0)}{dt}=1$ (3.22)

The general solution of the eqn. (3.21) is:

i(t) = Complimentary function (C.F) + Particular solution (P.I)

The complimentary function is obtained as follows:

On solving auxiliary eqn. $m^2 + 2nm + n^2 = 0$ we get roots –n,-n then

$C.F = [c_1 + c_2 t]e^{-nt}$ (3.23)

And particular integral is given by

$P.I = \frac{1}{D^2 + 2nD + n^2}\{n^2 \sin nt\} = \frac{n}{2}\int \sin nt\, dt = -\frac{1}{2}\cos nt$ (3.24)

$i(t) = [c_1 + c_2 t]e^{-nt} - \frac{1}{2}\cos nt$ (3.25)

Using initial conditions in (3.22) we obtain $c_1 = \frac{1}{2}$, $c_2 = 1 + \frac{n}{2}$. Hence the exact solution of the eqn. (3.21) is

$i(t) = \frac{1}{2}[(1 + (2 + n)t)e^{-nt} - \cos nt]$ (3.26)

By using the Maclaurin's series of e^{-nt} and $\cos nt$ in eqn.(3.26) we get

$i(t) = t - nt^2 + (n^2/2 + n^3/6)t^3 - (n^3/6 + n^4/12)t^4 + O(5)$

(3.27)

Now applying the differential transformation on both sides of eqn. (3.21) we get

$\frac{(k+2)!}{k!} I(k+2) + 2n\frac{(k+1)!}{k!} I(k+1) + n^2 I(k) =$

$n^2 \frac{n^k}{k!} \sin(k\frac{\pi}{2})$ (3.28)

By using equations (2.1) and (3.22) the following transformed conditions at t = 0 can be obtained:

$I(0) = 0, I(1) = 1$ (3.29)

We can rewrite equation (3.28) as:

$I(k+2) =$

$\frac{1}{(k+1)(k+2)}[$

$-2n(k+1)\,I(k+1) - n^2\,I(k) + \frac{n^{k+2}}{k!}\sin(k\frac{\pi}{2})]$

(3.30)

For k = 0, 1, 2 we get:

$I(2) = -n$

$I(3) = \frac{1}{6}[3n^2 + n^3]$

$I(4) = -\frac{1}{12}[2n^3 + n^4]$ (3.31)

By the inverse differential transformation eqn. (2.3) we have

$i(t) = \sum_{k=0}^{N} I(k)\,(t - t_0)^k$

k = 0, 1, 2, 3, 4

$= I(0) + I(1)t + I(2)t^2 + I(3)t^3 + I(4)t^4$

$= t - nt^2 + \frac{1}{6}[3n^2 + n^3]t^3 - \frac{1}{12}[2n^3 + n^4]t^4$

which is same as (3.27).

3. CONCLUSIONS

In this study the differential transformation method is successful in solving linear differential equations arising in electrical circuit problems. Three equations are solved. The method gives the exact solutions in series form. For higher order of approximation with a greater degree of accuracy more computations must be needed. This method is an important tool in solving linear differential equations with constant coefficients with minimum size of computations.

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