Rabi Frequency Estimation In Simulated Oscillating Qubit Dynamics Using Autocorrelation-based Linear Regression Model

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Abstract – *An autocorrelation-based linear regression model is used for parameter (i.e. Rabi frequency) estimation in a simulated oscillating pure qubit state dynamics. In a comparison with the Bayesian approach for Rabi frequency estimation, it is shown that both approaches have good convergence but the autocorrelation-based approach converges slower.*

Key Words: **Rabi Frequency, Qubit State, Hamiltonian, Bloch Vector, Regression, Estimation, Autocorrelation.**

1. INTRODUCTION

Quantum state monitoring and control are crucial for the development of quantum-based technologies [1], [2], thus state and parameter estimation are essential. State and parameter estimation also form a vital component of robust control systems design. In [3] it is shown that as long as the qubit Hamiltonian parameters are accurately modeled, the state estimation always converges to the true qubit state. In the case of modeling errors in the Hamiltonian, the qubit state estimation will not necessarily converge to the true state [4] hence it is necessary to augment the parameter estimation problem to the state estimation problem. Qubit state tracking and control problem has been addressed in many articles including but not limited to [3], [5], [6], [7].

In this paper, we simulate a real qubit evolution and use unsharp measurement to periodically measure the state of the qubit. The aim then is to develop a qubit state estimator to estimate the state of the real qubit in real-time. We assume our estimator starts its evolution not only with the wrong qubit state but also with the wrong Rabi frequency in its Hamiltonian. The goal is now to estimate both the correct Rabi frequency and the evolving real qubit state. In [4] the Bayesian estimator is used to tackle this very same problem and shown to be successful. In this paper, we use an autocorrelation-based linear regression model to tackle this estimation problem and compare its performance to that of the Bayesian approach used in [4].

In [8], [9] we presented a frequency estimation problem formulated as a linear regression problem based on the differential [8] and autoregressive [9] relationship in the periodic data. In this paper, we adopt the autoregressive [9] approach in formulating the Rabi frequency estimation problem as a linear regression problem.

The rest of this paper is organized as follows. Section 2 gives some background on qubit dynamics. Section 3 presents the dynamics of an undriven underdamped second-order system and its autocorrelation function which is adopted as the baseline model for frequency estimation. The autocorrelation-based autoregressive model is used to assemble a linear regression cost function. A brief description of the Bayesian estimation approach is given in this section. Section 4 presents simulation results for the models presented in Section 3 and gives a discussion on these results. Section 5 concludes this work with a summary of major findings and some remarks.

2. SIMULATED QUBIT DYNAMICS

A two-level system (i.e. qubit) undergoing Rabi oscillations will evolve under the following Hamiltonian in a rotating frame [3],

$$\widehat{H}_R = \hbar\left(\frac{\Omega_R}{2}\right)\hat{\sigma}_x \quad (1)$$

With Ω_R as the Rabi frequency and $\hat{\sigma}_x$ as the Pauli matrix responsible for rotations about the x-axis on the Bloch sphere. The positive operator-valued measure (POVM) measurements are done periodically on the qubit and the measurement outcomes are used to update the estimators. The i^{th} POVM measurement on a qubit with state $|\psi_i\rangle$ will result in a post-measurement state $|\psi_{i+1}\rangle$ given by,

$$|\psi_{i+1}\rangle = \frac{\widehat{M}_{ni}|\psi_i\rangle}{\sqrt{p(x_i|\psi_i)}} \quad (2)$$

where a conditional probability $p(x_i|\psi_i)$ is given by,

$$p(x_i|\psi_i) = \langle\psi_i|\widehat{M}_{ni}^{\dagger}\widehat{M}_{ni}|\psi_i\rangle \quad (3)$$

with $\widehat{M}_{ni}$ as the Krauss operator corresponding to unsharp measurement [3]. In the absence of measurement, the qubit state evolves under the unitary operator U_t given by,

$$\widehat{U}_t = e^{-\frac{i\widehat{H}_R}{\hbar}t} \tag{4}$$

If the measurement is carried out regularly after every time interval T, the unnormalized post-measurement state after N measurement will be given by,

$$|\tilde{\psi}_N(t = NT)\rangle = \widehat{M}_{nN}\widehat{U}_T \ldots \widehat{M}_{n2}\widehat{U}_T\widehat{M}_{n1}\widehat{U}_T|\psi_0(t = 0)\rangle \tag{5}$$

The outcome of each i^{th} measurement can be considered as either an up or down Bloch vector deflection (i,e, $x_i = \pm\delta$) in the z-basis. The probability of a qubit following this trajectory is given by the inner product of the post-measurement state equation (5) above as,

$$p(x_N|\psi_N(t = NT) = \langle\tilde{\psi}_N(t = NT)|\tilde{\psi}_N(t = NT)\rangle \tag{6}$$

This probability will be useful in Bayesian estimation to determine which Rabi frequency (among many) is likely to have produced the sequence of measurement outcomes $\{x_i\}_{i=1}^{N}$.

3. RABI FRENQUECY ESTIMATION MODELS

3.1 Differential Dynamics

The autocorrelation function $y(\tau)$ of some periodic data $\{x_i = \pm\delta\}_{i=1}^{N}$ from measurement can be modeled by the following damped oscillation response [10],

$$y(\tau) = Ae^{-\xi\omega_0\tau}\cos\left(\omega_0\tau\sqrt{1-\xi^2} + \theta\right) + b \tag{7}$$

which satisfies the following second-order homogeneous ODE [11],

$$y'' + \frac{\omega_0}{Q}y' + \omega_0^2 y = 0 \tag{8}$$

with A as the amplitude, b as the signal offset, ω_0 as the angular frequency, θ as the phase offset, the parameter $\xi = (2Q)^{-1}$ as the damping ratio and τ as the time lag.

3.2 Autoregressive Dynamics

Using centered differencing with a discretization time parameter h, equation (8) is approximated discretely as [9],

$$y_{i+1} + \alpha y_i + \beta y_{i-1} = 0 \tag{9}$$

with α and β as the coefficients to be determined in a linear regression problem. The next section presents the linear regression problem based on this autoregressive model in equation (9) above.

3.3 Autoregressive Linear Regression Model

In [9] it is shown that given the function points or data $y_i(\tau_i)$ of size N (i.e. $i = 1, 2, 3, \ldots, N$) from an oscillatory process, the frequency estimation problem can be posed as a linear regression problem shown below,

$$J(\alpha,\beta) = \sum_{i=2}^{N-1}(y_{i+1} + \alpha y_i + \beta y_{i-1})^2 \tag{10}$$

whose solution is given by,

$$\alpha = \frac{\sum_{i=2}^{N-1} y_{i-1}y_i \sum_{i=2}^{N-1} y_{i-1}y_{i+1} - \sum_{i=2}^{N-1} y_{i-1}^2 \sum_{i=2}^{N-1} y_i y_{i+1}}{\sum_{i=2}^{N-1} y_{i-1}^2 \sum_{i=2}^{N-1} y_i^2 - \left(\sum_{i=2}^{N-1} y_{i-1}y_i\right)^2} \tag{11}$$

$$\beta = \frac{\sum_{i=2}^{N-1} y_{i-1}y_i \sum_{i=2}^{N-1} y_i y_{i+1} - \sum_{i=2}^{N-1} y_i^2 \sum_{i=2}^{N-1} y_{i-1}y_{i+1}}{\sum_{i=2}^{N-1} y_{i-1}^2 \sum_{i=2}^{N-1} y_i^2 - \left(\sum_{i=2}^{N-1} y_{i-1}y_i\right)^2} \tag{12}$$

which gives the estimated frequency and damping ratio as,

$$\omega_0 = \frac{1}{h}\sqrt{\frac{2(1+\alpha+\beta)}{1+\beta}} \tag{13}$$

$$\xi = \frac{1-\beta}{\sqrt{2(1+\beta)(1+\alpha+\beta)}} \tag{14}$$

3.4 Bayesian Estimation Model

In [4], a population of equally-spaced possible frequencies Ω_i $(i = 1,2,3,\ldots,k)$ is kept in a set and each frequency is used to evolve an associated qubit state estimator. This means there are as many qubit state estimators as there are frequencies. On each measurement, the probability $p_i(n|\psi)$ of each estimator is computed, evolved independently according to the Bayesian update rule, and the posterior probabilities are shown in a plot against frequencies. As more measurements are made, the estimator whose frequency is closest to the true Rabi frequency will have the highest posterior probability and is taken as the estimated Rabi frequency. This is closely related to how a particle filter works.

This Bayesian estimator converges to the correct Rabi frequency provided the exact Rabi frequency is present in the list/population of discrete frequency estimators. In the case whereby the exact Rabi frequency is not present in the population set (i.e. the exact Rabi frequency lies between two adjacent estimators), the Bayesian estimator can keep oscillating between these two closest frequencies. It is, therefore necessary that the frequency spectrum be finely discretized (or interpolation methods adopted) to reduce the quantization error in frequency estimation. This however calls for a large set of frequency estimators hence there is an inherent trade-off between estimation accuracy and computational intensity. More details on this can be found in [4].

4. QUBIT SIMULATION RESULTS & DISCUSSION

We set up our simulation experiment to run for 20000 time-steps with a true qubit evolving from an initial qubit state $|\psi_0\rangle = [1\ 0]^T$ under the evolution operator $\widehat{U}_T$ described in equation (4) with the normalized Rabi frequency $\Omega/\Omega_R = 1/3$. All estimators described in the sections below were initialized with the state $|\psi_{E0}\rangle = [0\ 1]^T$ which is orthogonal to the true state $|\psi_0\rangle$. This is essential for testing qubit state estimation on top of Rabi frequency estimation. The Krauss operators used for the i^{th} unsharp measurements are,

$$\widehat{M}_{0i} = diag([\sqrt{p_0} \quad \sqrt{1-p_0}]) \tag{15}$$

$$\widehat{M}_{1i} = diag([\sqrt{1-p_0} \quad \sqrt{p_0}]) \tag{16}$$

with $p_0 = 0.3120$ indicating the measurement strength and ten measurements were made per Rabi cycle. We simulate the i^{th} unsharp measurement using a random number $0 \leqslant r_i \leqslant 1$ and the conditional probability $p_i = p(x_i|\psi_i) = \langle\psi_i|\widehat{M}_{0i}^{\dagger}\widehat{M}_{0i}|\psi_i\rangle$. If $r_i < p_i$ we record the measurement outcome as $x_i = -\delta$ and use $\widehat{M}_{0i}$ to obtain the post-measurement state as described in equation (2). Otherwise if $p_i \leqslant r_i$ we record the measurement outcome as $x_i = +\delta$ and use $\widehat{M}_{1i}$ to get the post-measurement state.

4.1 Autocorreleration-based Rabi Frequency Estimator

Fig. 1 below shows the graph of the autocorrelation function $y(\tau)$ computed from the unsharp measurement outcomes $\{x_i\}_{i=1}^N$. An autoregressive linear regression model was used to estimate the Rabi frequency and the damping ratio which were then used to simulate an estimated autocorrelation function shown superimposed on the original $y(\tau)$ in Fig. 1.

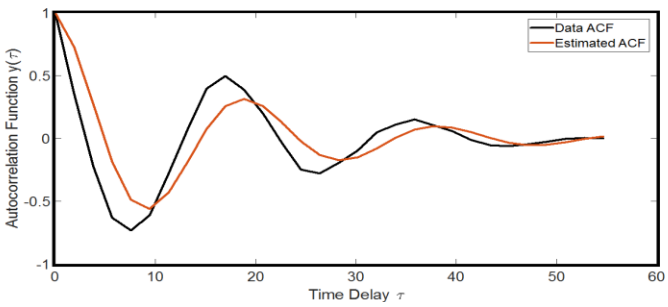

Fig. 1 *Autocorrelations from measurement data and simulated data based on estimated Rabi frequency.*

4.2 Bayesian Rabi Frequency Estimator

Fig. 2 below shows a plot of 101 frequency estimates along with their respective probabilities. Each probability indicates the likelihood of the respective frequency estimate to have been responsible for the qubit evolution that led to the sequence of measured outcomes $\{x_i\}_{i=1}^N$.

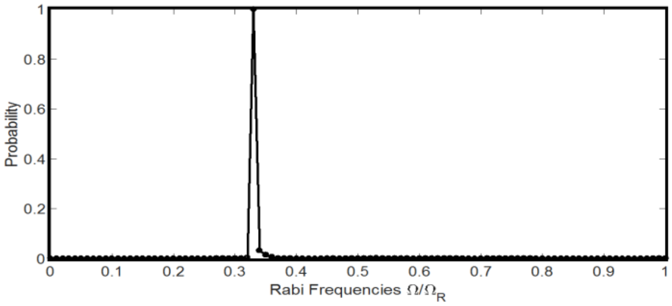

Fig. 2 *Population of Bayesian-evolved Rabi frequency estimates and their likelihood to be responsible for the sequence of measurements obtained.*

4.3 Convergence of Rabi Frequency Estimators

Fig. 3 below shows the convergence of the two Rabi frequency estimators. Despite its nonsmooth convergence profile, the Bayesian estimator is relatively quick to converge to the true Rabi frequency. The autocorrelation-based estimator shows a rather smooth but slow asymptotic convergence towards the true Rabi frequency.

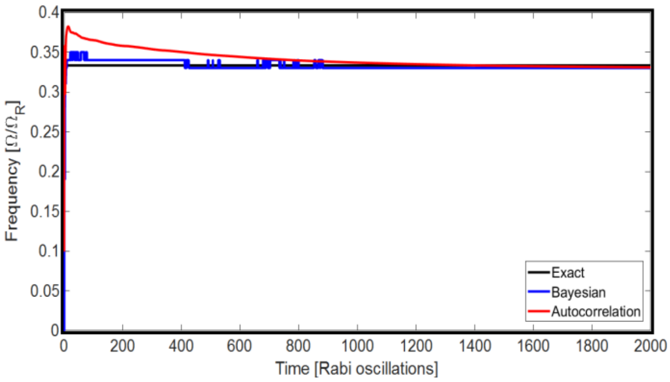

Fig. 3 *Convergence of Bayesian Rabi frequency estimator and autocorrelation-based Rabi frequency estimator.*

The true Rabi frequency (normalized) being a non-ending decimal like 3^{-1} makes it difficult for the Bayesian estimator (formulated as a fixed/static grid of frequency population) to converge to the exact Rabi frequency. This is because the true Rabi frequency falls between some two fixed consecutive frequency estimators thus there will always be a quantization/discretization error, in estimation, quantified by the discretization in the frequency spectrum.

It is interesting to note that as the measurement unsharpness is increased towards half (i.e. $p_0 \to 0.5$) the performance of autocorrelation-based Rabi frequency estimation drops significantly faster than that of Bayesian Rabi frequency estimation. One advantage of the Bayesian estimation approach is that it is quick to respond to changes in the parameter being estimated while the autocorrelation-based estimation is slow due to the averaging effect inherent in the autocorrelation computation. In the case of multiple parameter estimation, the Bayesian approach can be computationally intensive due to multidimensional discretizations.

4.4 Convergence of Qubit State Estimators

Fig. 4 below shows the convergences (in terms of the fidelities) of the two qubit state trajectories estimated by the Bayesian estimator and the autocorrelation-based estimator. On each i^{th} measurement, the fidelity F_i between the true qubit state $|\psi_i\rangle$ and the estimator state $|\psi_{Ei}\rangle$ was computed as follows,

$$F_i = \|\langle\psi_i|\psi_{Ei}\rangle\|^2 \quad (17)$$

The fidelity F_i in equation (17) above quantifies an overlap between the two states and it takes values in the range of $0 \leqslant F_i \leqslant 1$, with $F_i = 0$ indicating no overlap (i.e. 0% overlap) while $F_i = 1$ indicates 100% overlap between the two states.

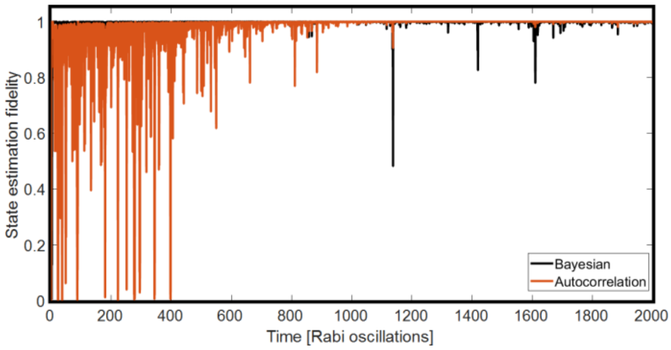

Fig. 4 *Fidelities for Bayesian qubit state estimator and autocorrelation-based qubit state estimator.*

Overall, both estimators show good convergence towards the true qubit state trajectory. It can be observed here also that the autocorrelation-based qubit state estimator is relatively slower to converge to 100% fidelity.

5. CONCLUSIONS

We have successfully shown how the Rabi frequency estimation problem can be approached from the autocorrelation of unsharp measurements and solved as an autoregressive linear regression problem. This autocorrelation-based Rabi frequency estimation is compared with the Bayesian Rabi frequency estimation presented in the literature. It was found that both estimators have a good convergence for the unsharp measurement strength of $p_0 = 0.3120$ both estimators have good convergence with the Bayesian Rabi frequency estimator outperforming the autocorrelation-based Rabi frequency estimator by faster convergence to the true Rabi frequency.

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