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Population Modelling – Exponential and Logistic Models

Nalin Dewan[1]

[1]*Student, Dept of Information Technology, Delhi Technological University, New Delhi, India*

Abstract - *We cannot have a sustainable planet without stabilizing population. As the world's population grows, so does the need for resources such as water, land, forests, and energy. Unfortunately, other endangered plants, animals, and natural resources pay the price for all of this "increase and demand" in an increasingly volatile and dangerous climate. This demands the development of a mathematical model capable of accurately forecasting future population growth rates and population statistics. Mathematical models, as one of the languages of science, may predict the behavior of systems based on physics, chemistry, biology, and other disciplines. Certain mathematical models may be used to accurately forecast economic and social systems, including population increase. The current work focuses on population growth mathematical modelling utilizing exponential and logistic growth models, which are nothing more than differential equations that allow us to examine population size changes over time and estimate the population of a given location at a given time. The prediction is compared to the actual population of the past, based on a model that accurately forecasts population growth rates and may be used to predict future population growth rates.*

Key Words: Mathematical modelling, Population growth, Exponential model, Logistic model, Growth rate, Differential equations.

1.INTRODUCTION

The projection of a country's population is important for planning and decision-making in terms of socioeconomic and demographic development. The world's most pressing concern now is the enormous population growth. The growing human population footprint must be taken into account in any truly meaningful conservation and sustainability efforts.

Every day, hundreds of thousands of people are added to the world's population, each of whom need adequate land, water, shelter, food, and energy to live a decent life. In order to improve people's living conditions in terms of fundamental requirements such as food, water, education, and health care, the society must plan and implement appropriate measures in a time bound manner.

Mathematical modelling is a broad interdisciplinary science that uses mathematical and computational techniques to model and elucidate events in the life sciences, with the goal of predicting behaviour that is similar to that observed in the real world.

It involves the following processes.

1. The mimicking of a real-world problem in mathematical terms: thus, the construction of mathematical model.
2. The analysis or solution of the resulting mathematical problem.
3. The interpretation of the mathematical results in the context of the original situation.

Modelling the growth of a species appears to be difficult at first, because the population of any species changes in integer quantities. As a result, no species' population can ever be a differentiable function of time. But population is a large quantity hence, when a large population is abruptly expanded by one unit, the difference is insignificant in comparison to the initial population.

As a result, we approximate the large populations in such a way that they change continuously and are even differentiable with time. Future population projections are usually based on current population and a credible growth rate, and the growth of various species is governed by first order differential equations. Furthermore, unrestrained exponential growth is not feasible in any species since the population and growth rate are constrained by the numerous variables that define the species' sustainability and growth rate.

2. THE EXPONENTIAL GROWTH MODEL

This model was proposed by Thomas R. Malthus proposed in 1798. The main assumption of the model is that population growth rate is directly proportional to the size of population.

2.1 Assumptions of the Model

The Malthus model is one of the most basic growth models for any reproducing population; however, it is far too basic to be effective in most situations. I it assumes that the population is homogenous (i.e., all members are identical), that there is an infinite supply of nutrients, that there are no geographical constraints, that the population lives in a uniform and unchanging environment, and that growth is density-independent. Population expansion is constrained by a variety of reasons, ranging from resource availability to predation. Internal system dynamics, such as overpopulation, impose additional constraints. For a limited period of time, the Malthus model may accurately describe population growth; nevertheless, unrestricted growth is never sustainable, and so other components are required to get a more realistic model.

2.2 Mathematical Calculations

Supposing we know the population P_0 at some given time $t=t_0$, and we are interested in protecting the population P, at some future time $t=t_1$,

i.e., to find the population function P(t) for $t_0 < t < t_1$ that satisfies $P(t_0)=P_0$

Malthus gave the equation

$$P(t+h) = P(t) + bhP(t) - dhP(t)$$

Here I used the fact that the total population at the moment t+h can be found as the total population at the moment t plus the number of individuals born during time period h minus the number of died individuals during time period h. b and d are per capita birth and death rates respectively (i.e., the numbers of births and deaths per one individual per time unit respectively). From the last equality I find

$$\frac{P(t+h) - P(t)}{P(t)} = (b-d)P(t)$$

Next, I *postulate* the existence of the derivative

$$\frac{dP}{dt} = \lim_{h \to 0} \frac{P(t+h) - P(t)}{h}$$

We assume for simplicity that both *b* and *d* are constant, and hence obtain an ordinary differential equation

$$\frac{dP}{dt} = (b-d)P(t)$$

This can be written as

$$\frac{dP}{dt} = kP(t) \qquad (1)$$

Where $t_0 < t < t_1$

$$lnP = kt + C$$

$$P = P_0 e^{k(t-t_0)} \qquad (2)$$

2.3 Geometrical Analysis

Assume equation,

$$N(t) = N_0 e^{rt}$$

Malthusian growth described by Equation 1 can manifest as both exponential growth and exponential decay (Figure 1); for instance, exponential growth occurs for all (Figure 2); however, reversing the sign of r, the model becomes one in which a population decays exponentially in time as the fraction r of individuals is removed per unit time (Figure 3).

Viewing the phase line (Figure 1), the linear rate of change in population size, or density, is portrayed for both exponential growth and exponential decay. The only equilibrium solution

$\frac{dN}{dt} = 0$ occurs when $N(t) = 0$ Qualitatively, we can judge the equilibrium point's stability based on whether trajectories approach, $+\infty$ for all $r > 0$ or zero for all $r < 0$.

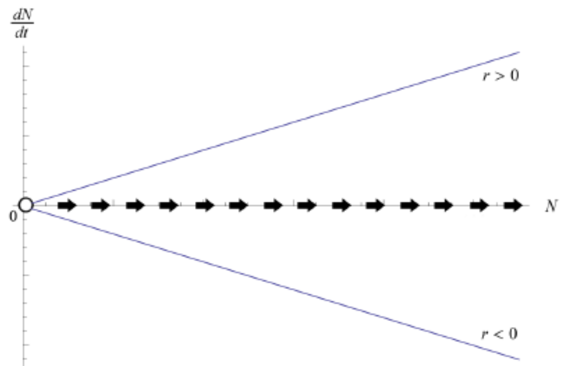

Figure 1: Phase line portrait of exponential model given by Equation1: phase trajectory reveals the linear rate of change in growth as a function of N.

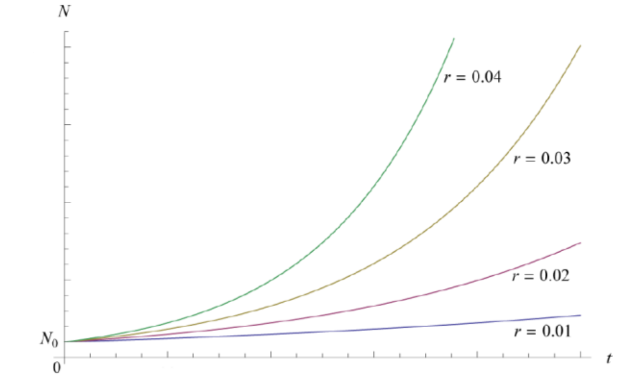

Figure 2: Dynamics of exponential growth given by Equation 1: exponential growth for a set of arbitrary positive growth rates r.

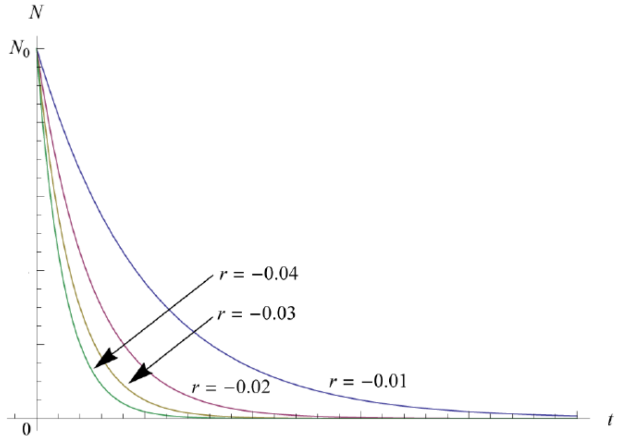

Figure 3: Dynamics of exponential growth described by Equation 1 exponential decay for a set of arbitrary negative growth rates r.

Where N(t) is the population size at the time moment t, k is a constant known as the Malthus factor and is equal to (b-d), is the multiple that determines the growth rate.

For any t, we can reasonably state $P(t) > 0$. If $k > 0$, the population will continue to grow. According to Equation (1) population growth increases as $P(t)$ increases. In other words, as the population grows, so does the growth rate. Equation (1) is suitable for modelling population growth under ideal conditions; nevertheless, we must acknowledge that a more realistic model must take into account the reality that a specific ecosystem has finite resources.

2.4 Accuracy of the Model

Let us express equation 2 as

$$f(t, a, b) = ae^{bt}$$

We can see the system for the partial Derivatives is no longer linear. Hence it cannot be easily solved without applying specific numerical procedures.

$$\frac{\partial f}{\partial a} = 0 \text{ and } \frac{\partial f}{\partial b} = 0$$

One can avoid this obstacle by noticing that log *f* is a linear function of *a* and *b*:

$$\log f(t, a, b) = \log a + bt = A + bt$$

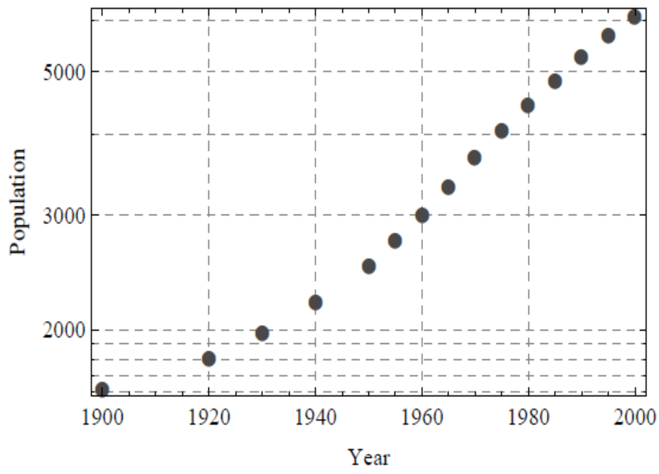

Figure: The total world population during the 20th century, millions versus years, in logarithmic coordinates

This transformation is also often useful for presenting the data in logarithmic coordinates

$$f(t, a, b) = 1.15 \times 10^{-9} e^{0.01462t}$$

$$f(a, b) = 1600000$$

which is better than the linear approximation but worth than the quadratic one, see the graphical comparison.

Based on the data of census and available estimates of population of the last 2000 years we find that the exponential model is highly inaccurate.

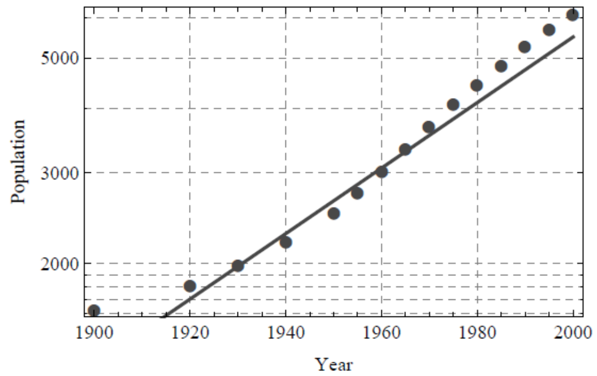

Figure: The total world population during the 20th century, millions versus years, and the best exponential approximation of the data, in logarithmic coordinates.

I used the logarithmic coordinates to plot the population numbers. I.e., instead of *P(t)* I plotted log *P*(*t*). If the population growth was exponential then the graph obtained should have been a straight line which it was not, proving inaccuracy of Malthus's Model.

A better initial value problem could be of hyperbolic form

$$P(t) = \frac{C}{T-t}$$

Where $C \approx 2 \times 10^{11}, T \approx 2026$ (can be found out by the method of least squares)

This formula is very precise if I consider only 400-500 years of the population estimates up to 1960. Note that when $t \to T$, the population *blows up*. Hence this is also inaccurate.

3. THE LOGISTIC GROWTH MODEL

Verhulst, a Belgian mathematician, showed that population growth is dependent not only on the size of the population, but also on how far it is from its upper limit, or carrying capacity (maximum supportable population). He modified Malthus's model to make the population size proportionate to the previous population as well as a new term that's

$$\frac{a - bP(t)}{a}$$

Where a and b are called the vital coefficients of the population. This term reflects how far the population is from its maximum limit

3.1 Assumptions of the Model

The logistic model is based on the same assumptions as the Malthusian model, with the exception that the reproduction rate is proportionate to the population size when the population is small and negatively proportional when the population is high. The carrying capacity is the K at which the population converges, and the parameter K is defined conceptually.

However, as the population value grows and gets closer to $\frac{a}{b}$, this new component decreases until it reaches zero, providing the necessary feedback to restrict population increase.

Thus, the second term models the competition for available resources, which tends to limit the population growth.

3.2 Mathematical Calculations

$$\frac{dP(t)}{dt} = \frac{aP(t)(a - bP(t))}{a}$$

$$\frac{dP(t)}{dt} = aP(t) - b(P(t))^2$$

We can assume, $K = \frac{a}{b}$ this is the *carrying capacity*.

$$\frac{dP(t)}{dt} = aP(t)\left(1 - \frac{P(t)}{K}\right) \qquad (3)$$

The growth of population in above equation is no longer infinite, but will reach a maximum value because it is limited by growth limiting factors $\left(1 - \frac{P(t)}{K}\right)$.

Growth models that have these characteristics are logistic growth models that are built based on logistical principles. This rule leads to the notion that the population will approach the equilibrium point at a point when the supply of logistics is very low. Because the number of births and deaths in the population are expected to be equal at this time, the graph of its function will be almost constant.

$$\int \frac{dP}{P(1 - \frac{P(t)}{K})} = \int a\, dt$$

$$= \int \frac{dP}{P(K-P)} = \int \frac{a}{K}\, dt$$

Which results in

$$P(t) = \frac{K}{1 + \left(\frac{K}{P_0} - 1\right) e^{-at}}$$

where is P(t) the population at time t, P_0 is the initial population (at time $t = 0$), i.e., $P(0) = P_0$ and k is the carrying capacity (based on the logistical supplies).

Another form of the equation is

$$P(t) = \frac{KP_0}{P_0 + (K - P_0)e^{-at}}$$

or

$$P(t) = \frac{KP_0 e^{-at}}{P_0 e^{-at} + (K - P_0)} \qquad (4)$$

$for\ t \to \infty\ or\ k > 0$, the maximum population size is $P_{max} = \lim_{t\to\infty} P(t) = K$.

3.2.1 Obtaining Equilibrium Points

To obtain equilibrium points

$$\frac{dP(t)}{dt} = 0$$

Which means either

$$aP(t) = 0 \Rightarrow P(t) = 0$$

Or

$$1 - \frac{P(t)}{K} = 0 \Rightarrow P(t) = K$$

Thus, the logistic equation has exactly two equilibrium points.

3.2.2 Geometrical Analysis

Viewing the phase line in Figure 1 we can discern a number of facts concerning the system's dynamics; in fact, we see that the equilibria are already obtained graphically. We can also observe that any point P_0 on the trajectory will

$\boldsymbol{P(t)}$	$\boldsymbol{\frac{dP(t)}{dt}}$
$\boldsymbol{P > K}$	$\frac{dP(t)}{dt} < 0$
$\boldsymbol{0 < P < K}$	$\frac{dP(t)}{dt} > 0$
$\boldsymbol{P = K}$	$\frac{dP(t)}{dt} = 0$
$\boldsymbol{P = 0}$	$\frac{dP(t)}{dt} = 0$

Table: Behaviour logistic growth, described by Equation 3 for different cases of N

approach $K\ ans\ t \to \infty$ with the exception of the case $P_0 = 0$ where there is no population.

$$\frac{d^2P}{dt^2} = \frac{a^2(K - P_0)(K - P_0 - P_0 e^{at})}{(K - P_0 + P_0 e^{at})} P(t)$$

The equilibrium at $P_0 = 0$ is unstable as $\frac{d^2P}{dt^2} < 0$ and $P_0 = K$ represents stable equilibrium as $\frac{d^2P}{dt^2} > 0$, where P asymptotically approaches the carrying capacity K.

$$P_{max} = \lim_{t\to\infty} P(t) = K\ \ where\ P_0 > 0$$

A point of inflection occurs at $P = \frac{K}{2}$ for all solutions that cross it, and we can see graphically that growth of P is rapid (graph has upward concavity) until it passes infection point $P = \frac{K}{2}$ From there, subsequent growth slows (graph has downward concavity) as P asymptomatically approaches K.

At point of inflection

And

$$P = \frac{K}{2}$$

$$t_F = \frac{1}{a} log\,(\frac{K - P_0}{P_0})$$

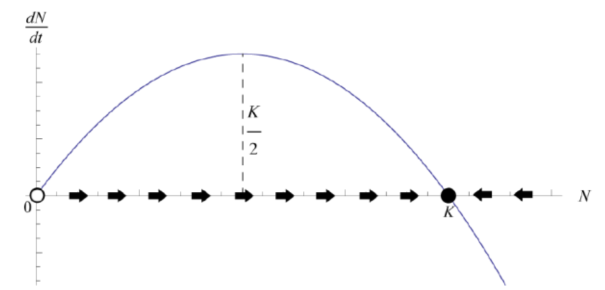

Figure 4: Phase line portrait of logistic growth, as described by Equation 3.

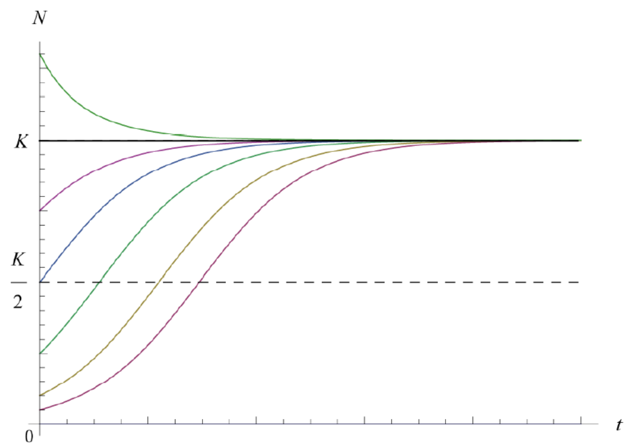

Figure 5: Dynamics of the logistic model given by Equation 3

3.2.3 Maximum Specific Growth Rate and Lag Time

The maximum specific growth rate μ is the slope of the line tangent to the curve at the inflection point t_0

$$\mu = \frac{dP(t)}{dt}\bigg|_{t=t_0}$$

And the lag time λ denotes the intercept with the t-axis of this tangent line.

For the logistic model the maximum specific growth rate is given by,

$$\mu = \frac{P_0 K(K-P_0)ae^{-at_F}}{(P_0+(K-P_0)e^{-at_F})^2}$$

Where,

$$t_F = \frac{1}{a}log\left(\frac{K-P_0}{P_0}\right)$$

$$\mu = \frac{Ka}{4}$$

I obtained a value of maximum specific growth rate and it is independent from the choice of the initial size P_0.

The line tangent to the logistic curve at its inflection point is represented by,

$$p = \mu t + \frac{K}{2} - \frac{K}{4}\log\left(\frac{K-P_0}{P_0}\right)$$

The lag time of the logistic model is,

$$\mu\lambda + \frac{K}{2} - \frac{K}{4}\log\left(\frac{K-P_0}{P_0}\right) = 0$$

$$\mu\lambda = \frac{K}{4}\log\left(\frac{K-P_0}{P_0}\right) - \frac{K}{2}$$

$$\lambda = \frac{1}{\mu}\left(\frac{K}{4}\log\left(\frac{K-P_0}{P_0}\right) - \frac{K}{2}\right)$$

Put $\mu = \frac{Ka}{4}$ in above equation,

$$\lambda = \frac{1}{a}\log\left(\frac{K-P_0}{P_0}\right) - \frac{2}{a}$$

We can notice that the lag time is positive if, and only if,

$$\left(\frac{K-P_0}{P_0}\right) > e^2$$

$$0 < P_0 < \frac{K}{1+e^2}$$

3.2.4 Threshold Crossing

The time that the value of the population size spends below (or above) a specific threshold denoted by S.

For a growth model $P(t)$, we want to analyse time instant θ when $P(t)$ crosses S, given that $S > P(0)$.

If we consider bounded populations, the threshold S represents a percentage p of the carrying capacity K

$$S = pK \ with \ P(0) < S < K$$

Which can be written as,

$$\frac{P_0}{K} < p < 1$$

$$P(\theta) = \frac{KP_0}{P_0+(K-P_0)e^{-a\theta}} = S = pK$$

$$p = \frac{P_0}{P_0 + (K - P_0)e^{-a\theta}}$$

$$\theta = \frac{-1}{a}\log\left(\frac{P_0}{K - P_0}\right)\frac{-1}{a}\log\left(\frac{1-p}{p}\right)$$

$$\theta = t_F - \frac{1}{a}\log\left(\frac{1-p}{p}\right)$$

We can examine if, and when, the time instant θ precedes the inflection point t_F. Clearly, it depends on p.

We can conclude,

- When $1 < p < \frac{1}{2}$ then $\theta < t_F$.
- When $p = \frac{1}{2}$ then $\theta = t_F$
- When $\frac{1}{2} < p < 1$ then $\theta > t_F$

3.2.5 Calculating K without 'a'

If put time t=1 and t=2, then the values of P are P_1 and P_2 respectively, from equation 4 we get,

$$\frac{1}{K}(1 - e^{-a}) = \frac{1}{P_1} - \frac{e^{-a}}{P_0} \qquad (A)$$

$$\frac{1}{K}(1 - e^{-2a}) = \frac{1}{P_2} - \frac{e^{-2a}}{P_0} \qquad (B)$$

Dividing equation (B) by (A) to eliminate $\frac{1}{K}$ we get,

$$(1 + e^{-1}) = \frac{\frac{1}{P_2} - \frac{e^{-2a}}{P_0}}{\frac{1}{P_1} - \frac{e^{-a}}{P_0}}$$

$$e^{-a} = \frac{P_0(P_2 - P_1)}{P_2(P_1 - P_0)}$$

Substituting value of e^{-a} from equation (A),

$$K = \frac{P_1(P_0P_1 - 2P_0P_2 + P_1P_2)}{P_1^2 - P_0P_2}$$

3.3 Accuracy of Model

Again, using the available data and the method of the least squares, I can estimate the three parameters of the logistic curve, and find, e.g., that K = 11740, that is, the world population will stabilize at approximately 12 billion people, see the comparison of the data with the best logistic fit in the figure below.

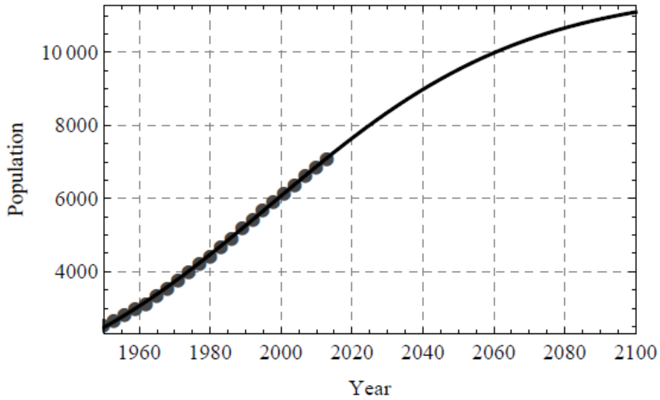

Figure: World population versus time, in millions, for the last 65 years and the best logistic fit together with prediction increase of the world population of the world population

3.4 Some variations to Logistical Model

3.4.1 Logistics Growth Model with Time -Dependent Carrying Capacity

Carrying capacity K in equation 4 is modelled as a function of time and the amount is assumed to be more than zero, or the initial carrying capacity is $k_1 > 0$. The growth of the carrying capacity of the environment will stop when it reaches its maximum, for example k_2.

These assumptions cause the growth equation that is influenced by the carrying capacity of the environment to be the logistical equation 2 where $K(t)$ satisfies the logistical equation 3

$$\frac{dK(t)}{dt} = aK(t)\left(1 - \frac{K(t)}{K}\right)$$

In this model it is assumed that the carrying capacity is also modelled with a logistical model that is identical to equation 3.

The population at one time is assumed to be more than 0 ($k_1 > 0$) and its growth will stop when it reaches the maximum population k_2. Using this assumption obtained a population growth model which is a modification of equation 3 with

$$N(t) = (K(t) - k_1)$$

and

$$K = k_2$$

Or,

$$\frac{dK(t)}{dt} = a_1(K(t) - k_1)\left(1 - \frac{(K(t) - k_1)}{k_2}\right) \qquad (5)$$

with a_1 is intrinsic growth rate. The particular solution of equation 5 is,

$$K(t) = \frac{k_1 + k_1e^{-a_1t} + k_2}{1 + e^{-a_1t}}$$

Or,

$$K(t) = k_1 + \frac{k_2}{1 + k_2 e^{a_1 t}}$$

If $a_1 > 0$ then for $t \rightarrow \infty$ obtained $K(t) = k_1 + k_2$, it means that the saturation point for growth in carrying capacity is when $K(t) = k_1 + k_2$.

By substituting equation (7) to (3), obtained the rate of population growth with environmental carrying capacity as a function of time as follows

$$\frac{dP(t)}{dt} = aP(t)\left(1 - \frac{P(t)(1 + e^{-a_1 t})}{k_1 + k_1 e^{-a_1 t} + k_2}\right)$$

$$\frac{dP(t)}{dt} = \frac{(k_1 + k_1 e^{-a_1 t} + k_2)P_0}{(2k_1 + k_2 - 2P_0)e^{at} + P_0(1 + e^{-a_1 t})}$$

Calculating Equilibrium Points,

$P_1 = 0$

And,

$$P_2 = \frac{k_1 + k_1 e^{-at} + k_2}{1 + e^{-at}}$$

$if\ t \rightarrow \infty\ and\ K(t) = k_1 + k_2$, max population is obtained,

$$P_{max} = \frac{k_1 + k_1 e^{-a_1 t} + k_2}{2(1 + e^{-a_1 t})}$$

Maximum growth rate is,

$$\left(\frac{dP(t)}{dt}\right)_{max} = \frac{a(k_1 + k_1 e^{-a_1 t} + k_2)}{4(1 + e^{-a_1 t})}$$

3.4.2 Logistic Harvesting Model with Time-Dependent Carrying Capacity

Let $j(P)$ be the rate of growth with the carrying capacity of a time-dependent environment as follows

$$j(P(t)) = aP(t)\left(1 - \frac{P(t)(1 + e^{-a_1 t})}{k_1 + k_1 e^{-a_1 t} + k_2}\right)$$

Let us assume r(N) is the rate of harvest with α is the effort of the harvest rate. Effort of harvest rate is a positive constant.

$$r(N) = \alpha P$$

The rate of population growth with time-dependent harvesting is defined as the rate of population growth with environmental carrying capacity $j(P)$ which is affected by the rate of harvesting with time dependent $r(P)$. So, the harvesting model is

$$\frac{dP(t)}{dt} = j(P) - r(P)$$

$$\frac{dP(t)}{dt} = aP(t)\left(1 - \frac{P(t)(1 + e^{-a_1 t})}{k_1 + k_1 e^{-a_1 t} + k_2}\right) - \alpha P(t)$$

Calculating Equilibrium Points,

$P_1 = 0$

And,

$$P_2 = \frac{(a - \alpha)}{a}(k_1 + k_2)$$

Maximum population with harvesting is,

$$P_{max} = \frac{(a - \alpha)(k_1 + k_1 e^{-a_1 t} + k_2)}{2a(1 + e^{-a_1 t})}$$

Maximum growth rate is,

$$\left(\frac{dP(t)}{dt}\right)_{max} = \frac{(a - \alpha)^2(k_1 + k_1 e^{-a_1 t} + k_2)}{4a(1 + e^{-a_1 t})}$$

4. EXPONENTIAL VS LOGISTIC MODELS – WHICH IS BETTER?

I will use data by UN World Population Prospectus (2019) to model population. I will predict population based on both exponential and logistic models and match with actual data by United Nations.

Note: The data Provided by the UN is in thousands.

4.1 Exponential Model

We have,

$$P = P_0 e^{k(t - t_0)}$$

Where $t = 1970, t_0 = 1960$,

$P_0 = 3034950$ and P $= 3700437$

$$k = \frac{1}{10}\ln\left(\frac{3700437}{3034950}\right)$$

$k = 0.01982559709$

Average growth rate per year is 1.982559709

$$P = 3034950 \times e^{0.01982559709(t-1960)}$$

4.2 Logistic Model

As we have data of years in multiples of ten, we consider 10 years as 1 unit.

$$P_{max} = K$$

$$K = \frac{P_1(P_0P_1 - 2P_0P_2 + P_1P_2)}{P_1^2 - P_0P_2}$$

Where,

$P_0 = 3034950$ (Population in 1960)

$P_1 = 3700437$ (Population in 1970)

$P_2 = 4458003$ (Population in 1980),

$P_{max} = K = 15116431.3$

$$a = \frac{1}{t}\ln\left(\frac{\frac{K}{P_0} - 1}{\frac{K}{P(t)} - 1}\right)$$

To calculate a, we use

$t = 1$, $P_0 = 3034950$, $P(t) = 3700437$ and $K = 15116431.3$

By putting these values in above equation, we get,

$$a = 0.254914399$$

Which means growth rate in ten years is 25.4914399% or average growth rate per year is 2.54914399 %

$$P(t) = \frac{15116431.3}{1 + \left(\frac{15116431.3}{3034950} - 1\right)e^{-0.254914399t}}$$

The data after calculations is tabulated in table below: -

Year	Actual Population	Exponential Model	Logistic Model
1960	3034950	3034950	3034950
1970	3700437	3700437	3700437
1980	4458003	4511848.298	4458008.999
1990	5327231	5501181.364	5298675.558
2000	6143494	6707449.896	6205578.649
2010	6956824	8178222.301	7154587.435
2020	7794799	9971497.52	8116533.235
2030	8548487	12157992.18	9060629.945
2040	9198847	14823929.26	9958314.684

*The data beyond 2020 in Actual population is the estimate provided by the UN.

5. CONCLUSIONS

I have explained both exponential and logistic models in detail, and have later compared both of them with real time data from the United Nations. Based on my calculations, I have concluded, that the Logistic model is much more accurate that the exponential counterpart. The carrying capacity of the world is 15116431.3 thousand and growth rate is approximately 25.49% as calculated by the Logistic Model. Based on this model we also found out that the population of the world is expected to be 9958314.684 in the year 2040.

REFERENCES

1. Di Crescenzo A, Paraggio P. Logistic Growth Described by Birth-Death and Diffusion Processes. *Mathematics*. 2019; 7(6):489. https://doi.org/10.3390/math7060489
2. A. Tsoularis, J. Wallace, Analysis of logistic growth models, Mathematical Biosciences, Volume 179, Issue 1, 2002, Pages 21-55
3. 11. Bailey, N.T.J. The Elements of Stochastic Processes with Applications to the Natural Sciences; Reprint of the 1964 Original; Wiley: New York, NY, USA,
4. B. M. Al-Eideh and H. O. Al-Omar, "Population Projection Model using Exponential Growth Function with a Birth and Death Diffusion Growth Rate Processes," European Journal of Scientific Research, vol. 151, no. 3, (2019), pp. 271-276.
5. Windarto, Eridani and U. D. Purwati, "A new modified logistic growth model for empirical use," Commun. Biomath. Sci., vol. 1, no. 2, **(2018)**,
6. Ruly Budiono, Hafizan Juahir, Sukono, Mohamad Nurzaman, Mustafa bin Mamat, Agung Prabowo. (2020). Logistic Growth Model and Harvesting Factors. International Journal of Advanced Science and Technology, 29(06), 3310 -. Retrieved from http://sersc.org/journals/index.php/IJAST/article/view/14070
7. M. H. R. Doust and M. Saraj, "The Logistic Modelling Population: Having Harvesting Factor,"

Yugoslav Journal of Operation Research, vol. 25, no. 1, **(2015)**, pp. 107-115.
8. C. P. Ho and C. H. Lin, "A New Approach to the Logistic Modelling Population: Having Harvesting Factor," Yugoslav Journal of Operation Research, vol. 26, no. 1, **(2016)**, pp. 110-121.
9. Malthus, T. (1798). An Essay on the Principle of Population, as it Affects the Future Improvement of Society with Remarks on the Speculations of Mr. Godwin, M. Condorcet, and Other Writers (Vol. 6, p. 340). London.
10. Kingsland, S. E. (1985). Modeling nature: Episodes in the history of population ecology (p. 267). Chicago: University of Chicago Press.
11. Venkatesha P., G.Blessy Sachy Eunice, Akshaya B., Arya Kumari S. "Mathematical Modelling of Population Growth". International Journal of Science, Engineering and Management (IJSEM) Vol 2, Issue 11, (2017)