Autoregressive Linear Regression Model for Frequency Estimation

Naleli Jubert Matjelo[1]

[1]*National University of Lesotho, Department of Physics and Electronics, P. O. Roma 180, Lesotho*

Abstract – *In this paper, a frequency estimation problem is reformulated as a linear regression problem based on the autoregressive model relating the periodic data to its past instances. This frequency estimation approach, based on the autoregressive relation in data, is shown to be less prone to noise compared to the approach based on the differential relation in data. Incorporating the autocorrelation function of the periodic data is shown to improve the noise-rejection feature of this frequency estimation approach.*

***Key Words*: Autoregression, Linear Regression, Noise Rejection, Frequency Estimation, Autocorrelation.**

1. INTRODUCTION

State and parameter estimation is central to signal processing, control systems design, and other fields which make use of system identification and characterization techniques [1], [2], [3]. Frequency estimation is one example of parameter estimation which finds its use in many disciplines of Science including Physics [4], [5], Engineering [5], [6], Finance, and Economics [7].

In [8] we presented a frequency estimation problem formulated as a linear regression problem based on the differential relationship in the periodic data. This approach was shown to be prone to noise due to the inherent differential operators in the model formulation which amplify noise. In this paper, we present one way to model the frequency estimation problem as a linear regression problem based on the autoregressive (rather than differential) relationship in the periodic data.

The rest of this paper is organized as follows. Section 2 presents the dynamics of a free/unforced oscillation response model which is adopted as the baseline model for frequency estimation. Section 3 gives a detailed account of formulating the frequency estimation problem as a linear regression problem based on an autoregressive model. Both the autoregressive model data and its autocorrelation function are used for frequency estimation, with the drawbacks of each option mentioned. Section 4 presents simulation results for the models in Section 3 and gives a discussion on these results. Section 5 concludes this work with a summary of major findings and some remarks.

2. A FREE DAMPED OSCILLATORY MODEL

2.1 Differential Dynamics

An unforced harmonic oscillation can be modeled by the following homogeneous ODE as shown below [8],

$$y'' + \frac{\omega_n}{Q} y' + \omega_n^2 y = 0 \tag{1}$$

with its solution taking the following form,

$$y(t) = Ae^{-\xi\omega_n t}\cos\left(\omega_n t\sqrt{1-\xi^2} + \theta\right) + b \tag{2}$$

where A is the amplitude, b is the signal offset, θ is the phase offset and the parameter $\xi = (2Q)^{-1}$ is the damping ratio.

2.1 Autoregressive Dynamics

The dynamics in equation (1) above can be approximated as the difference equation in the discrete-time domain by adopting some discretization methods such as Runge-Kutta or Euler methods. Using centered differencing with a discretization time parameter h, equation (1) is approximated discretely as,

$$y_{n+1} - \frac{2-h^2\omega_n^2}{1+h\xi\omega_n} y_n + \frac{1-h\xi\omega_n}{1+h\xi\omega_n} y_{n-1} = 0 \tag{3}$$

which can also be written compactly as,

$$y_{n+1} + \alpha y_n + \beta y_{n-1} = 0 \tag{4}$$

with α and β as the coefficients in equation (3) above. The next section presents the linear regression problem based on the autoregressive model in equation (4) above.

3. REGRESSION-BASED FREQUENCY ESTIMATION

Given a data $y_i(t_i)$ of size N (i.e. $i = 1, 2, 3, \dots, N$) from an oscillatory process, we can formulate the problem of frequency estimation as a linear regression problem by adopting equation (4) as part of the regression cost function shown below,

$$J(\alpha, \beta) = \sum_{i=2}^{N-1} (y_{i+1} + \alpha y_i + \beta y_{i-1})^2 \tag{5}$$

The resulting optimal solution to this regression problem is given by,

$$\alpha = \frac{\sum_{i=2}^{N-1} y_{i-1}y_i \sum_{i=2}^{N-1} y_{i-1}y_{i+1} - \sum_{i=2}^{N-1} y_{i-1}^2 \sum_{i=2}^{N-1} y_i y_{i+1}}{\sum_{i=2}^{N-1} y_{i-1}^2 \sum_{i=2}^{N-1} y_i^2 - \left(\sum_{i=2}^{N-1} y_{i-1}y_i\right)^2} \quad (6)$$

$$\beta = \frac{\sum_{i=2}^{N-1} y_{i-1}y_i \sum_{i=2}^{N-1} y_i y_{i+1} - \sum_{i=2}^{N-1} y_i^2 \sum_{i=2}^{N-1} y_{i-1}y_{i+1}}{\sum_{i=2}^{N-1} y_{i-1}^2 \sum_{i=2}^{N-1} y_i^2 - \left(\sum_{i=2}^{N-1} y_{i-1}y_i\right)^2} \quad (7)$$

From which the frequency and damping ratio are estimated as,

$$\omega_n = \frac{1}{h}\sqrt{\frac{2(1+\alpha+\beta)}{1+\beta}} \quad (8)$$

$$\xi = \frac{1-\beta}{\sqrt{2(1+\beta)(1+\alpha+\beta)}} \quad (9)$$

As in [8] the time-series data $y_i(t_i)$ can be replaced by the autocorrelation function $z_i(t_i)$ of the data for robustness against noise.

4. SIMULATION RESULTS & DISCUSSION

The linear regression model presented above is simulated with both noiseless and noisy data in this section.

4.1 Frequency Estimation With Noiseless Data

4.1.1 Signal Data-based Frequency Estimation

Fig. 1 below shows the plot of data from which the frequency should be estimated with the regression model described in the previous section. The data was generated with the following parameters corresponding to equation(4); $A = 5\,V$, $b = 7\,V$, $\theta = 0.25\pi\ rads$, $\omega_n = 3\pi\ rad/s = 9.4248\ rad/s$, $\xi = (15\pi)^{-1} = 0.02122$.

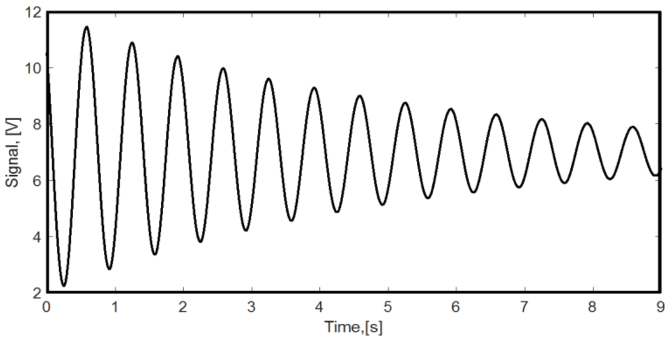

Fig. 1 Noiseless data.

In Fig. 2 below an offset estimate $b = N^{-1}\sum_{i=1}^{N} y_i \approx 6.9527\ V$ has been removed from the data.

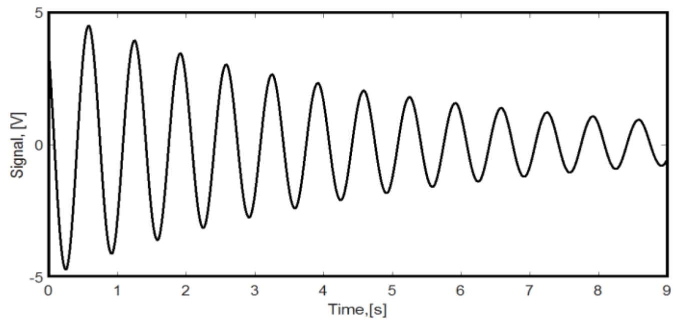

Fig. 2 Zero-mean noiseless data.

This offset estimate is 0.7% lower than the exact value of $b = 7.00$. Fig. 3 shows the first time advance (i.e. a set $\{y_{i+1}\}$) of the zero-mean data.

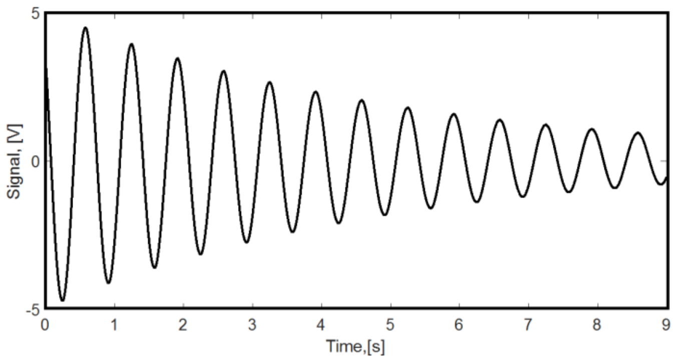

Fig. 3 The first derivatives of zero-mean data.

Fig. 4 below shows the second time advance (i.e. a set $\{y_{i+2}\}$) of the zero-mean data.

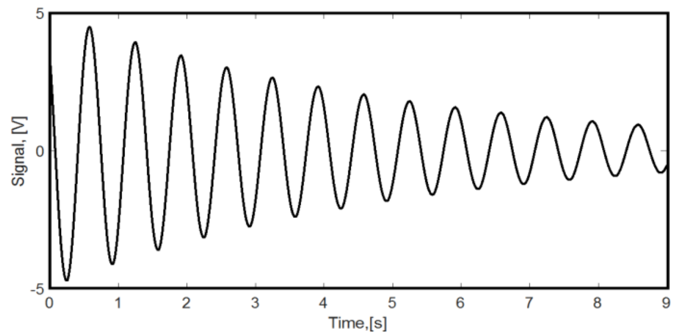

Fig. 4 The second derivatives of zero-mean data.

Fig. 5 shows the graph simulated based on estimates of frequency and damping ratio obtained from the data-based linear regression model outlined in the previous section. This graph is superimposed on the zero-mean data.

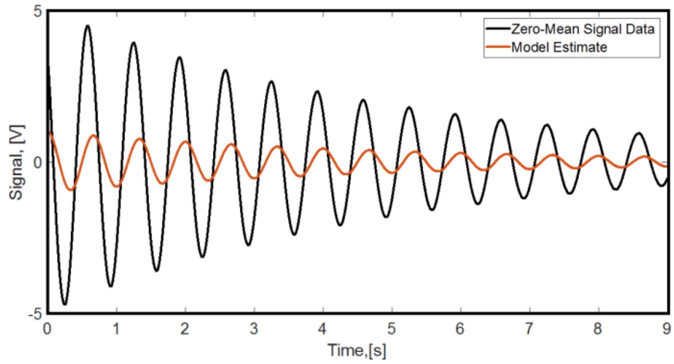

Fig. 5 Frequency and Q-factor estimates simulated.

The recovered frequency and damping ratio estimates are,

$$\omega_n \approx 9.4270\ rad/s \quad (10)$$

$$\xi \approx 0.02154 \quad (11)$$

with the estimated frequency being off by 0.023% from the exact value and the estimated damping ratio being off by 1.51% from the exact value.

4.1.2 Autocorrelation-based Frequency Estimation

In this section the oscillatory data $y_i(t_i)$ is replaced by its autocorrelation function $z_i(t_i)$ to carry out frequency estimation. Fig. 6 shows the plot of the zero-mean data and its autocorrelation function.

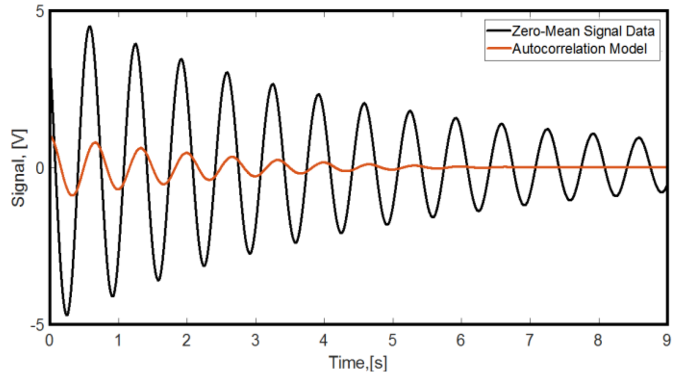

Fig. 6 Zero-mean data and its autocorrelation function.

Fig. 7 below shows the first time advance of the autocorrelation function shown in Fig. 6 above.

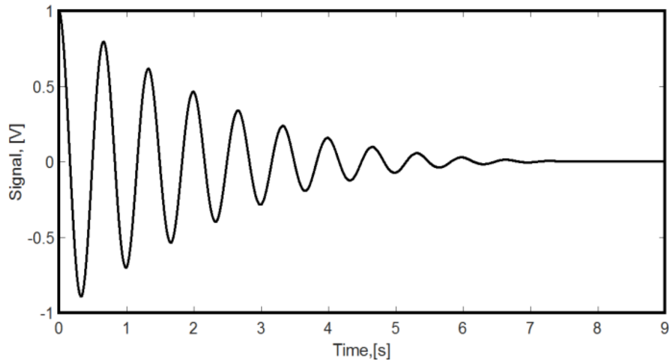

Fig. 7 Autocorrelation function's first time-advance.

Fig. 8 below shows the second time-advance of the autocorrelation function shown in Fig. 6 above.

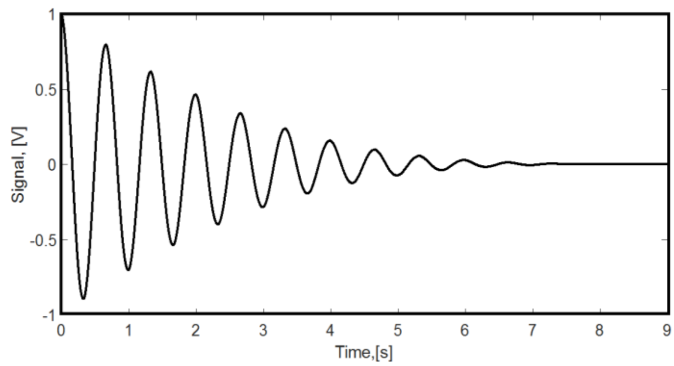

Fig. 8 Autocorrelation function's second time-advance.

Fig. 9 below shows the graph simulated based on frequency estimate and damping ratio estimate obtained from the autocorrelation-based linear regression model outlined in the previous section.

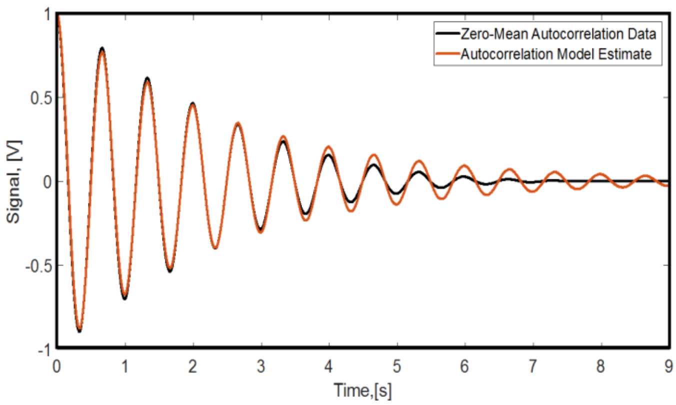

Fig. 9 Frequency and Q-factor estimates simulated.

The recovered frequency and damping ratio estimates are,

$$\omega_n \approx 9.4483\ rad/s \quad (12)$$

$$\xi \approx 0.04209 \quad (13)$$

with the estimated frequency being off by 0.25% from the exact value. The damping factor can be observed from the plot in Fig. 9 that the estimated damping ratio is smaller than that of the autocorrelation function based on the slower damping effect on the estimate compared to that seen on the autocorrelation function.

4.2 Frequency Estimation With Noisy Data

4.2.1 Signal Data-based Frequency Estimation

In this section, we look at the same frequency estimation approaches but here we subject them to the noise-infested signal data. The signal data has the same parameter setup as in the previous case except that here the is also an additive zero-mean white noise with an amplitude of 10 V.

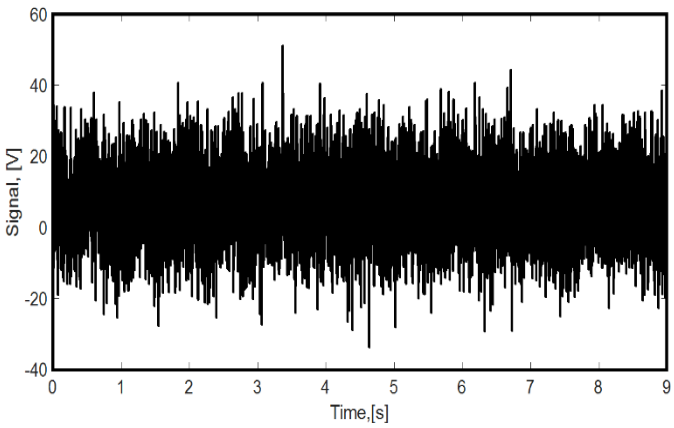

Fig. 10 The plot of noisy data.

Fig. 10 above shows the plot of this noisy signal data.

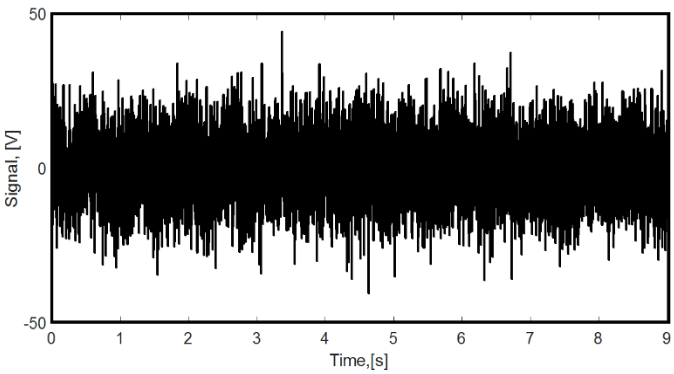

Fig. 11 Zero-mean noisy data.

Fig. 11 above shows the zero-mean of the noise data after removing the offset. The offset b was estimated as before and found to be 6.8179 V which is 2.6% off from the exact value. Fig. 12 below shows the first time advance of the noisy zero-mean data. Unlike in the differential formulation [8], Fig. 12 shows that the autoregressive formulation does not amplify noise judging by the magnitude of the original data in Fig. 11 and that of Fig. 12.

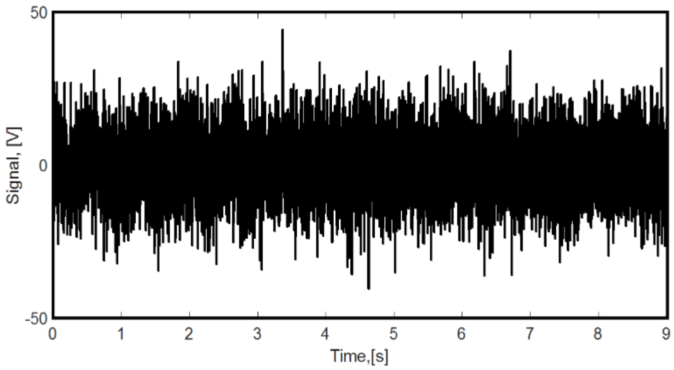

Fig. 12 The first time advance of zero-mean data.

Fig. 13 below shows the second time advance of the noisy zero-mean data. Again the noise is not amplified as compared to the case of differential formulation in [8].

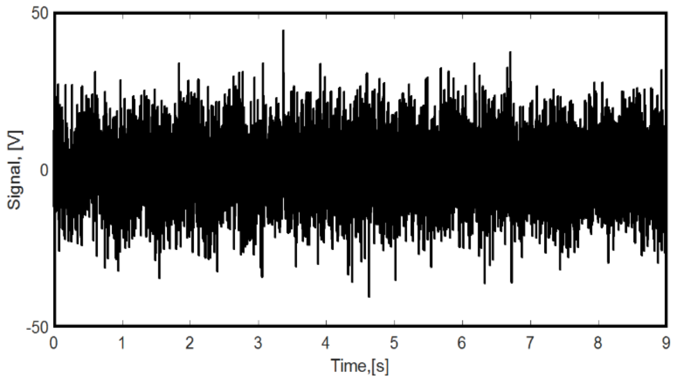

Fig. 13 The second time advance of zero-mean data.

Fig. 14 below shows the graph simulated based on the estimated values of frequency and damping ratio obtained from the noisy data used in the linear regression model outlined in the previous section. This graph is superimposed on the zero-mean data.

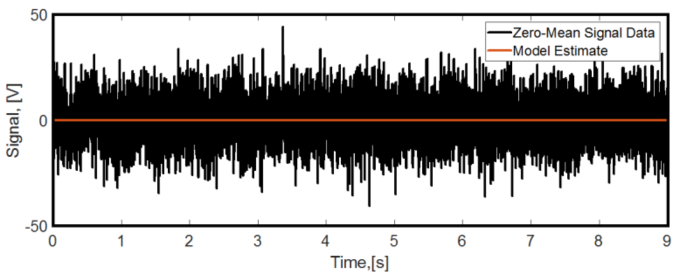

Fig. 14 Frequency and Q-factor estimate simulated.

It is clear that failed the model using noisy data fails to estimate both the frequency and damping ratio. The estimated frequency and damping ratio are given as,

$$\omega_n \approx 1403.8\ rad/s \quad (14)$$

$$\xi \approx 0.756 \quad (15)$$

which are both out by more than 100%. Despite this model failing to give good estimates under overwhelming noise, it is worth noting that it does not have the noise-amplification feature which seemed to be inherent in the differential formulation of the same problem [8]. Next, we look at how the autocorrelation based estimation model handle the same problem under the same noise conditions,

4.2.2 Autocorrelation-based Frequency Estimation

Fig. 15 below shows the plot of the autocorrelation of the noisy data superimposed on the plot of the noisy data itself.

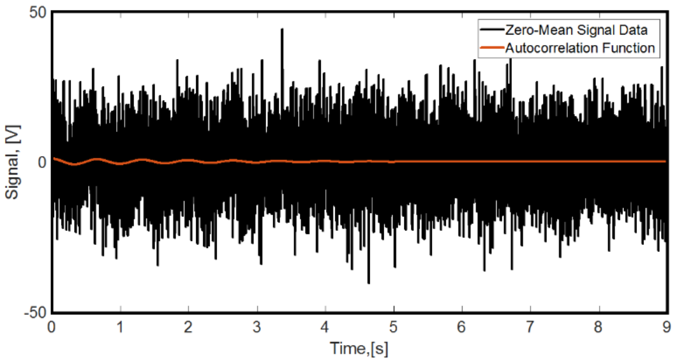

Fig. 15 Data and its autocorrelation function.

Fig. 16 shows the first time advance of the autocorrelation function of noisy data. As expected, both Fig 15 and Fig 16 plots show a significant noise-rejection or noise-attenuation due to the integrating effect inherent in the autocorrelation function operation.

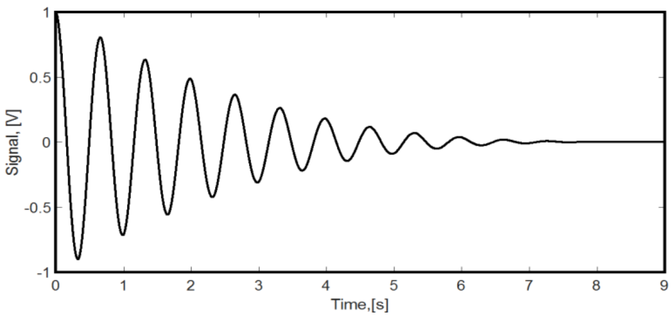

Fig. 16 Autocorrelation function's first time-advance.

Fig. 17 shows the second time-advance of the noisy data's autocorrelation function, looking practically noise-free.

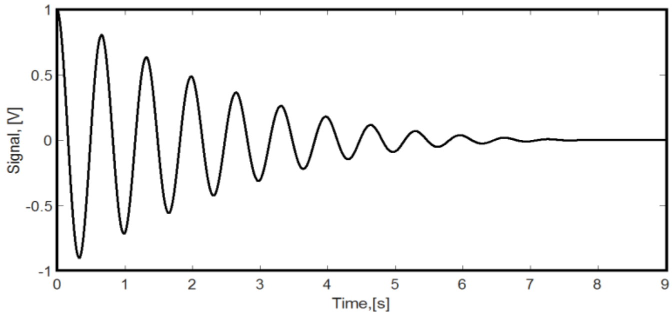

Fig. 17 Autocorrelation function's second time-advance.

Fig. 18 below shows the graph based on the estimated frequency and damping ratio using autocorrelation-based linear regression. This graph is superimposed on the autocorrelation function itself.

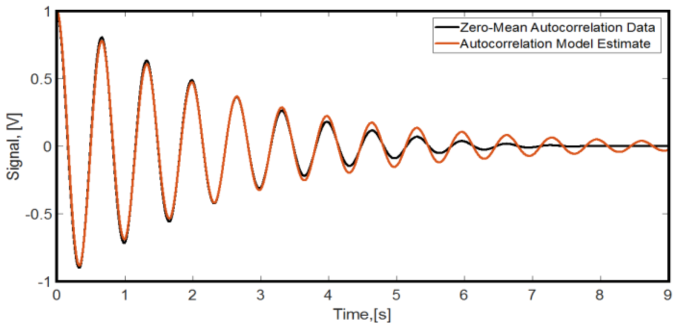

Fig. 18 Frequency and Q-factor estimate simulated.

The recovered frequency and damping ratio estimates are,

$$\omega_n \approx 9.4880\ rad/s \quad (16)$$

$$\xi \approx 0.0398 \quad (17)$$

with the estimated frequency being off by 0.67% from the exact value. This frequency estimation is still very good even given the condition of noise-infested signal data. The estimated damping factor can still be observed from the plot in Fig. 9 to be smaller than that of the autocorrelation function based on the slower damping effect on the estimate compared to that seen on the autocorrelation function. The amplitude and phase offset were not estimated in this work since the main focus was on estimating the frequency.

5. CONCLUSIONS

We have successfully shown how the frequency estimation problem can be posed as a linear regression problem based on the autoregressive relationship of the oscillatory data. It was shown that this model not as prone to noise as the model formulated based on the differential relationship in the oscillatory data. However, both formulations are not robust under overwhelming noise in the data. Replacing the oscillatory data with its autocorrelation function proved to be a robust approach against noise-infested data

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