Characterization and Estimation of Alpha Power Sujatha Distribution with Applications to Engineering Data

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Abstract - *This paper deals with a new version of Sujatha distribution, and we refer to the new distribution as alpha power transformed Sujatha distribution. For various parametric values, different shapes of the probability density function and hazard function have been provided. Some structural properties of the model have been derived and discussed. The estimation of the parameters is also carried out. At the end of the study, two data sets are examined to illustrate the usefulness and flexibility of the proposed model.*

***Keywords*-Alpha Power Transformation, Sujatha Distribution, Reliability Measures, Moments, Maximum Likelihood Estimation**

1. INTRODUCTION

Distribution fitting has played a significant role in many fields of science. It is used to choose a model which defines the structure of data resulting from a random process. The benefit of fitting distributions to real-life data is that more appropriate models for the random process can be developed. In statistical theory, extensions over the distributions have become a prominent method. Generally, new distributions with the addition of parameters can be obtained either by combining the existing distributions or by the use of generators. The objective of adding new parameters to the distribution provides more flexibility to the standard distributions for valuable analyses of complicated data structures. Sujatha distribution was suggested by Shanker [1] is a one-parameter continuous distribution with an increasing failure rate. This distribution is a mixture of three components, exponential distribution (θ), a gamma distribution (3,θ) and a gamma distribution (2,θ) with different mixing proportions.

To overcome the inadequacy of the existing distributions, several researchers introduced many generalizations and extensions of the Sujatha distribution. A two-parameter Sujatha distribution was presented by Tesfay and Shanker [2] and also discussed the different properties of the model. Size biased Lindley distribution and Sujatha distribution is a particular case of two-parameter Sujatha distribution. Shanker and Hagos [3] studied size-biased Poisson-Sujatha distribution and its applications to count data excluding zero counts. A generalization of Sujatha (AGS) distribution was developed by Shanker, Shukla *et al.* [4] in addition to some characteristics and it also illustrates its importance. Shanker, Shukla *et al.* [5] provided a new two-parameter Poisson-Sujatha (NTPPS) distribution. They investigated different model properties as well as their usefulness in discrete data. A new two-parameter Poisson-Sujatha distribution is a mixture of Poisson distribution which includes Poisson-Sujatha distribution and Poisson-Akash distribution as special cases. A two-parameter Poisson-Sujatha distribution was proposed by Shanker, Leonida *et al.* [6] with its characteristics along with the estimation of the parameters and demonstrate its usefulness by using real-life data. Shankar [7] introduced another two-parameter Sujatha distribution having a shape and scale parameters and studied its different properties. This distribution is a mixture of three components, a gamma distribution $(2,\theta)$, exponential distribution $(\theta), and\ a$ gamma $(3,\theta)$ distribution. New Quasi Sujatha distribution was proposed by Shanker and Shukla [8] and discussed its statistical characteristics, estimation and applicability is also determined by using real-life data.

Mahdavi and Kundu [9] suggested a model to develop a new family of distributions by incorporating a shape parameter using Alpha Power Transformation (APT) approach to bring more flexibility to the model. The model has the characteristic that it provides an upside-down-bathtub hazard function depending on its parameters. Alpha power transformed Aradhana distribution, Alpha power transformed Quasi Aradhana distribution, Alpha Power Garima distribution, and Alpha Power Rama distribution was introduced by Maryam and Kannan [10, 11, 12, and 13] for modeling real-life data. Some properties of the model were investigated and also illustrate the importance of the model.

In this study, a new distribution for modeling real-life data that provides more flexibility in data fitting than the existing distributions has been developed. In section 2, the Alpha power transformed Sujatha distribution has been presented. Reliability properties of the newly generated distribution has been obtained in section 3 and statistical properties in section 4. In section 5 and 6, entropy, order statistics has been studied. Bonferroni and Lorenz curves are discussed in section 7. The method of maximum likelihood and least square is employed for the parameter estimation in section 8. Finally, the new model is analyzed using the two data sets and a conclusion is presented in sections 9 and 10.

2. ALPHA POWER TRANSFORMED SUJATHA DISTRIBUTION

A random variable Y is said to have Sujatha distribution with parameter θ if its probability density function is given by

$$g\ (y;\theta) = \left(\frac{\theta^3}{\theta^2+\theta+2}\right)\ \left(1+y+y^2\right) e^{-\theta y}\ ; y>0,\ \theta>0 \quad (1)$$

and its cumulative distribution function (cdf) is given by

$$G(y;\theta) = 1-\left(1+\frac{\theta y\left(\theta y+\theta+2\right)}{\theta^2+\theta+2}\right) e^{-\theta y}\ ;\ y>0, \theta>0 \qquad (2)$$

Mahdavi and Kundu proposed the Alpha power transformation method (APT) to incorporate the additional (shape) parameter to the continuous distributions in order to increase their flexibility. For a given cumulative distribution function (cdf) and probability density function (pdf) of a baseline distribution, Mahdavi and Kundu defined the cumulative distribution function (cdf) of Alpha power transformation family of distributions as follows

$$F_{APT}\ (y;\alpha) = \begin{cases} \dfrac{\alpha^{G(y)}-1}{\alpha-1} & ;\ if\ \alpha \neq 1, \alpha>0,\ y \in R \\ G(y) & ;\ if\ \alpha = 1, \alpha>0,\ y\in R \end{cases} \qquad (3)$$

where, $G(y)$ and $g(y)$ are the cdf and pdf of the baseline distribution and the corresponding probability density function is given by

$$f_{APT}\ (y;\alpha) = \begin{cases} \left(\dfrac{\log \alpha}{\alpha-1}\right) g(y)\alpha^{G(y)} & ;\ if\ \alpha\neq 1, \alpha>0, y\in R \\ g(y) & ;\ if\ \alpha=1, \alpha>0, y \in R \end{cases} \quad (4)$$

The probability density function of the Alpha Power Transformed Sujatha (APS) distribution is defined from (4) by taking (1) and (2) and is given by

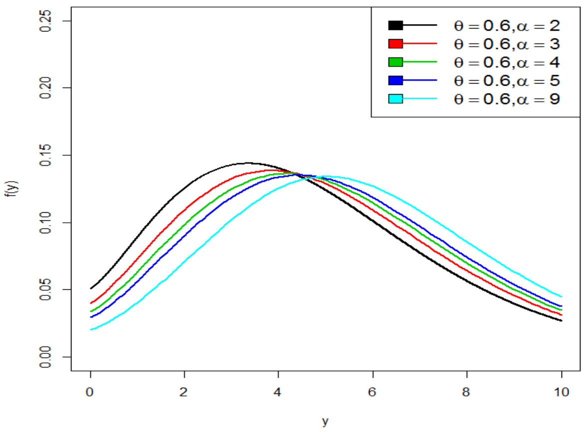
$$f_{APS}\,(y;\theta,\alpha) = \left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^3}{\theta^2+\theta+2}\right)\left(1+y+y^2\right)e^{-\theta y}$$

$$\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y}}\ ; y>0,\ \ \theta,\alpha>0, \alpha\neq 1 \qquad (5)$$

Substituting equation (2) into equation (3), the cumulative distribution function of Alpha Power Transformed Sujatha (APS) distribution is given by

$$F_{APS}\,(y;\theta,\alpha) = \frac{\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y}}-1}{\alpha-1}\ ;\ \ y>0,\ \ \theta,\alpha>0, \alpha\neq 1 \qquad (6)$$

The density function of the proposed distribution provide the desired shapes for different sets of parameters. The plots for pdf and cdf of the distribution for different values of the parameter (θ, α) is shown in figures 1 and 2.



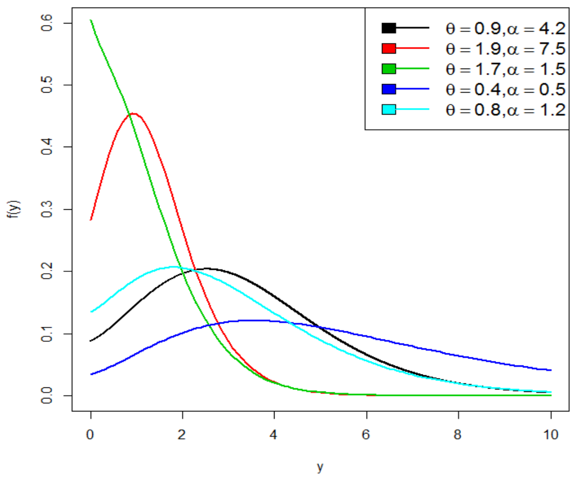


Fig- 1: Pdf plots of Alpha Power Sujatha distribution

Figure 1 reveals that the density function of the Alpha Power Transformed Sujatha distribution tends to the normal distribution when, $\alpha > 0$ and $\theta = 0.6$. The pdf is decreasing, left-skewed and uni-model for the parameter value $\alpha > 1$ and $\theta > 1$.

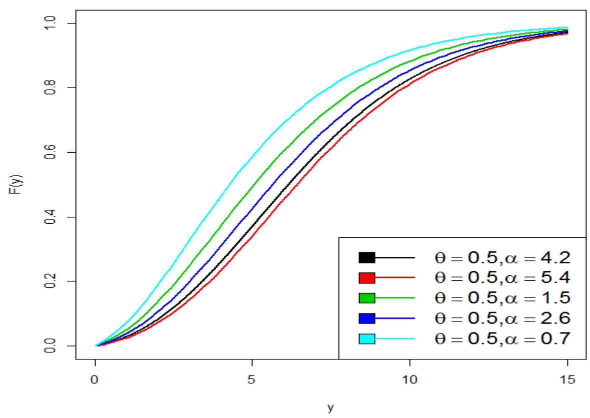


Fig- 2: cdf plot of Alpha Power Sujatha Distribution

3. RELIABILITY PROPERTIES

This section is devoted to obtain the expression for reliability measures of the model that include the reliability function, hazard function, mean residual life and mean waiting time of the Alpha power transformed Sujatha (APTS) distribution.

3.1 Reliability Function

The expression for reliability function of the Alpha power transformed Sujatha (APTS) distribution is expressed as

$$R(y;\theta,\alpha)=\frac{\alpha-\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y}}}{\alpha-1}; \quad \alpha\neq 1 \qquad (7)$$

3.2 Hazard Function

The expression for the hazard function of the Alpha power transformed Sujatha distribution is defined as

$$h(y;\theta,\alpha)=\frac{\alpha^{-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y}}}{1-\alpha^{-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y}}}\left(\frac{\theta^3}{\theta^2+\theta+2}\right) \qquad (8)$$

$$\left(1+y+y^2\right)e^{-\theta y}\log\alpha;\ \alpha>0,\alpha\neq 1$$

The plots of reliability function and hazard function of the distribution for different values of the parameter (θ, α) is presented in figure 3 and figure 4.

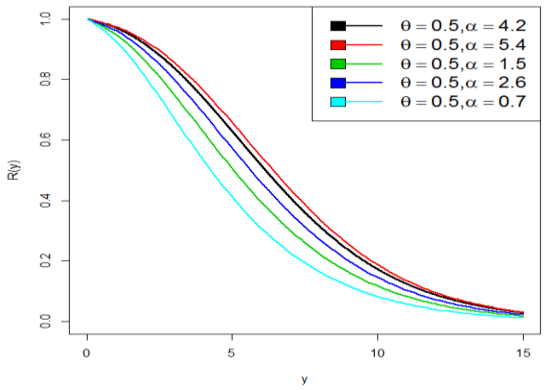


Fig- 3: Reliability plot of Alpha Power Sujatha Distribution

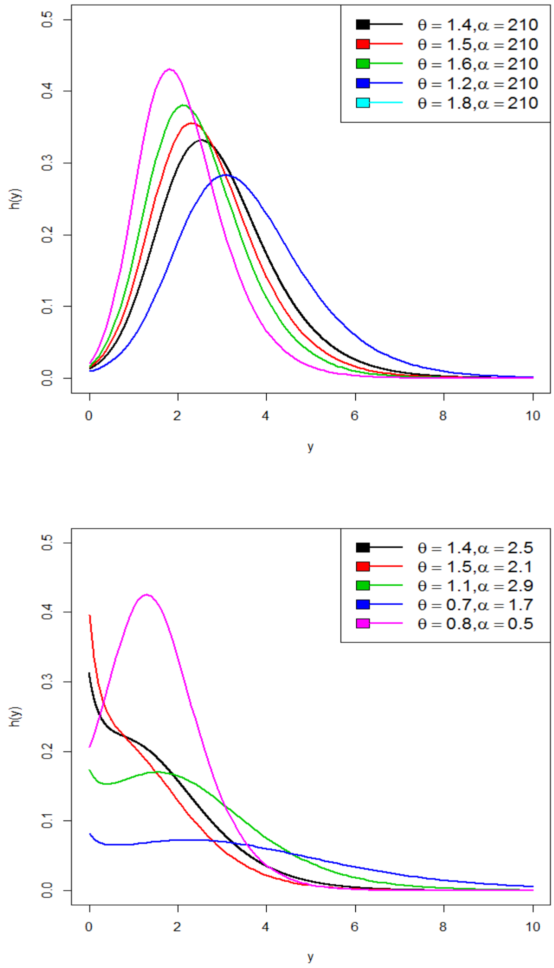


Fig- 4: Hazard plots of Alpha Power Sujatha Distribution

The graph displays various shapes for the hazard function of Alpha power transformed Sujatha distribution for different arbitrary parameter values. For different parameter values, the failure plot is left-skewed, uni-modal and it continues to decrease for, $\alpha > 0$ and $\theta > 0$.

3.3 Mean residual life

Let Y is a continuous random variable with reliability function (7). Then, the mean residual life of the Alpha power transformed Sujatha distribution is given by

$$\mu(t)=\frac{1}{S(t)}\left(E(t)-\int_0^t y f(y;\alpha,\theta)dy\right)-t \tag{9}$$

$$D=\int_0^t y f(y;\alpha,\theta)dy$$

$$D=\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^3}{\theta^2+\theta+2}\right)\int_0^t y\left(1+y+y^2\right)e^{-\theta y}\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y}}dy \tag{10}$$

Using the power series expansion to (10),

$$\alpha^{-y}=\sum_{r=0}^{\infty}\frac{(-\log\alpha)^r}{r!}y^r \tag{11}$$

The equation (10) becomes,

$$D=\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{\theta^3}{\theta^2+\theta+2}\right)\sum_{r=0}^{\infty}\frac{(-\log\alpha)^r}{r!}\int_0^t y\left(1+y+y^2\right)e^{-\theta y}\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)^r e^{-\theta r y}dy \tag{12}$$

Using the binomial expansion (13) to (12), we have

$$(1+y)^n=\sum_{s=0}^{n}\binom{n}{s}y^s \tag{13}$$

The equation (12) takes the form,

$$D=\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{\theta^3}{\theta^2+\theta+2}\right)\sum_{r=0}^{\infty}\sum_{s=0}^{r}\binom{r}{s}\frac{(-\log\alpha)^r}{r!}$$

$$\int_0^t y\left(1+y+y^2\right)\left(\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)^s e^{-\theta(r+1)y}dy$$

On simplification, the resulting equation becomes

$$\mu(t)=\frac{(\alpha-1)\left(\begin{array}{l}E(t)-\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\binom{r}{s}\binom{s}{t}\binom{t}{u}\left(\frac{1}{\theta^2+\theta+2}\right)^{s+1}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{(-\log\alpha)^r}{r!}\right)\left(\frac{2}{\theta}\right)^u\left(\frac{1}{r+1}\right)^{2s-t+2}\theta^{-t}\\ \left(\begin{array}{l}\left(\gamma(\theta(r+1))t,(2s-t+2)\right)+\left(\frac{1}{\theta(r+1)}\right)\left(\gamma(\theta(r+1))t,(2s-t+3)\right)\\ +\left(\frac{1}{\theta(r+1)}\right)^2\gamma\left((\theta(r+1))t,(2s-t+4)\right)\end{array}\right)\end{array}\right)}{\alpha-\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y}}}-t$$

3.4 Mean waiting time

The mean waiting time of random variable Y is defined by (14).

$$\bar{\mu}(t)=t-\frac{1}{F(t)}\int_0^t y\,f(y;\theta,\alpha)\,dy \tag{14}$$

Substituting equation (4) and equation (5) into equation (14), we obtain the mean waiting time of the Alpha power transformed Sujatha distribution as

$$\bar{\mu}(t)=t-\frac{\alpha-1}{\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y}}-1}\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\binom{r}{s}\binom{s}{t}\binom{t}{u}$$

$$\left(\frac{1}{\theta^2+\theta+2}\right)^{s+1}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{(-\log\alpha)^r}{r!}\right)\left(\frac{2}{\theta}\right)^u\theta^{-t}\left(\frac{1}{r+1}\right)^{2s-t+2}$$

$$\left(\begin{array}{l}\left(\gamma(\theta(r+1))t,(2s-t+2)\right)+\left(\frac{1}{\theta(r+1)}\right)\left(\gamma(\theta(r+1))t,(2s-t+3)\right)\\ +\left(\frac{1}{\theta(r+1)}\right)^2\gamma\left((\theta(r+1))t,(2s-t+4)\right)\end{array}\right)$$

4. STATISTICAL PROPERTIES

In this section, some characteristics of the distribution, such as moments, moment generating function, characteristic function, order statistics, renyi entropy, tsallis entropy, Bonferroni and Lorenz curve have been investigated.

4.1 Moments

Moments are of great importance to understand the most important characteristic of the distribution. They are helpful to get an idea about mean, variance, kurtosis and skewness of the lifetime data.

Let Y denotes a random variable with pdf (5) then the n^{th} order moment $E(Y^n)$ of APS distribution is defined as

$$E(Y^n) = \mu_n' = \int_0^\infty y^n f(y) dy$$

$$E(Y^n) = \left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^3}{\theta^2+\theta+2}\right)\int_0^\infty y^n\left(1+y+y^2\right)e^{-\theta y}\,\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y}} dy$$

(15)

Applying power series expansion (11) and binomial expansion (13) to equation (15), we get

$$E(y^n) = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{\theta^3}{\theta^2+\theta+2}\right)\sum_{r=0}^{\infty}\frac{(-\log\alpha)^r}{r!}$$
(16)
$$\int_0^\infty y^n\left(1+y+y^2\right)\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)^r e^{-\theta(r+1)y}dy$$

$$E(Y^n) = \left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{\theta^3}{\theta^2+\theta+2}\right)\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\binom{r}{s}\binom{s}{t}$$
$$\binom{t}{u}\left(\frac{\theta}{\theta^2+\theta+2}\right)^s\left(\frac{(-\log\alpha)^r}{r!}\right)\theta^s\left(\frac{2}{\theta}\right)^u \quad (17)$$
$$\int_0^\infty y^{2s+n-t}\left(1+y+y^2\right)e^{-\theta(r+1)y}dy$$

By using the gamma function

$$\gamma(\alpha,\beta) = \int_0^\infty y^{\beta-1}e^{-\alpha y}\,dy = \frac{\Gamma\beta}{\alpha^\beta}$$

to equation (17), and we obtain expression for the n^{th}moment of the Alpha power transformed Sujatha distribution.

$$E(Y^n) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\left(\frac{(-\log\alpha)^r}{(r-s)!(s-t)!(t-u)!u!}\right)\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{1}{\theta^2+\theta+2}\right)^{s+1}\left(\frac{2}{\theta}\right)^u$$
$$\left(\frac{(n+2s-t)!\,(\theta(r+1))^2+(n+2s-t+1)!\,(\theta(r+1))+(n+2s-t+2)!}{\theta^{(n-t)}(r+1)^{(n+2s-t+3)}}\right)$$

(18)

Taking the value for $n = 1$ in equation (18), we obtain the mean of APS distribution which is given by

$$E(Y) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\left(\frac{(-\log\alpha)^r}{(r-s)!(s-t)!(t-u)!u!}\right)\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{1}{\theta^2+\theta+2}\right)^{s+1}\left(\frac{2}{\theta}\right)^u$$
$$\left(\frac{(2s-n+1)!(\theta(r+1))^2+(2s-t+2)!(\theta(r+1))+(2s-t+3)!}{\theta^{(1-t)}(r+1)^{(2s-t+4)}}\right)$$

We get the second moment of APS distribution by plugging n=2 into equation (18).

$$E(Y^2) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\left(\frac{(-\log\alpha)^r}{(r-s)!(s-t)!(t-u)!u!}\right)\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{1}{\theta^2+\theta+2}\right)^{s+1}\left(\frac{2}{\theta}\right)^u$$
$$\left(\frac{(2s-n+2)!(\theta(r+1))^2+(2s-t+3)!(\theta(r+1))+(2s-t+4)!}{\theta^{(2-t)}(r+1)^{(2s-t+5)}}\right)$$

By plugging $n = 3$, we get the third moment of the APS distribution.

$$E(Y^3) = \sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\left(\frac{(-\log\alpha)^r}{(r-s)!(s-t)!(t-u)!u!}\right)$$
$$\left(\frac{1}{\theta^2+\theta+2}\right)^{s+1}\left(\frac{2}{\theta}\right)^u\left(\frac{\alpha\log\alpha}{\alpha-1}\right)$$
$$\left(\frac{(2s-n+3)!(\theta(r+1))^2+(2s-t+4)!(\theta(r+1))+(2s-t+5)!}{\theta^{(3-t)}(r+1)^{(2s-t+6)}}\right)$$

4.2 Moment Generating Function of APS Distribution

If Y be a random variable with cdf and pdf of Alpha power transformed Sujatha distribution, then the moment generating function of Y is defined by

$$M_Y(t) = E(e^{ty}) = \int_0^\infty e^{ty} f_{APS}(y)\,dy$$

Using the expansion, $e^{ty} = \sum_{p=0}^{\infty}\frac{(ty)^p}{p!}$, the above expression can be rewritten as

$$M_Y(t) = \sum_{p=0}^{\infty}\frac{t^p}{p!}\int_0^\infty y^p f_{APS}(y)dy$$

$$M_Y(t)=\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\sum_{p=0}^{\infty}\left(\frac{t^p}{p!}\right)\left(\frac{(-\log\alpha)^r}{(r-s)!(s-t)!(t-u)!u!}\right)$$
$$\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{1}{\theta^2+\theta+2}\right)^{s+1}\left(\frac{2}{\theta}\right)^u$$

$$\left(\frac{(p+2s-t)!(\theta(r+1))^2+(p+2s-t+1)!(\theta(r+1))+(p+2s-t+2)!}{\theta^{(p-t)}(r+1)^{(p+2s-t+3)}}\right)$$

4.3 Characteristic Function of APS distribution

If $Y \sim APSD$ then the characteristic function is obtained by using the relation.

$$\Phi_Y(t)=M_Y(it)$$

$$\Phi_Y(t)=\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\sum_{p=0}^{\infty}\left(\frac{(it)^p}{p!}\right)\left(\frac{(-\log\alpha)^r}{(r-s)!(s-t)!(t-u)!u!}\right)$$

$$\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{1}{\theta^2+\theta+2}\right)^{s+1}\left(\frac{2}{\theta}\right)^u$$
$$\left(\frac{(p+2s-t)!(\theta(r+1))^2+(p+2s-t+1)!(\theta(r+1))+(p+2s-t+2)!}{\theta^{(p-t)}(r+1)^{(p+2s-t+3)}}\right)$$

5. ENTROPY MEASURES

Clausius [14] provides an idea of entropy to measure the quantity of energy that does not generate energy in the system. Entropy has played an important role in many practices like physics, non-equilibrium processes, principle of thermodynamics and principle of maximum entropy production etc. Moreover, Shannon made of cornerstone upon which he laid out his theory of information and communication. Entropy and other associated measures play a vital role in current innovative approaches to artificial intelligence and collective behaviour.

Entropy or information theory is the study of the transmission, handling, utilization, and retrieval of information. Shannon [15] originally introduced the concept of entropy in his article, "Mathematical theory of communication" after viewing information as the resolution of uncertainty. Information is a collection of possible messages with the aim of sending them over a noisy channel and then having a receiver reconstruct the message with a low chance of error despite the noise. Entropy is an important measure used in the quantification, storage and transmission of information. There exist various entropy measures in literature but the most popular and widely used are given as

5.1 Renyi Entropy

Let Y be a random variable with cdf (4) and pdf (5) of Alpha power transformed Sujatha distribution, then the renyi entropy [16] of the proposed distribution is defined by

$$R(\delta)=\frac{1}{1-\delta}\log\int_0^\infty f^\delta(y)\,dy,\ \delta>0,\quad \delta\neq 1 \quad (19)$$

Substituting equation (5) into equation (19), (19) is rewritten as

$$R(\delta)=\frac{1}{1-\delta}\log\left(\begin{array}{l}\left(\frac{\log\alpha}{\alpha-1}\right)^{\delta}\left(\frac{\alpha\theta^3}{\theta^2+\theta+2}\right)^{\delta}\\ \int_0^\infty(1+y+y^2)^\delta e^{-\theta\delta y}\alpha^{-\delta\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y}}dy\end{array}\right) \quad (20)$$

Using the expansions (11) and (13) to equation (20), we get,

$$R(\delta)=\frac{1}{1-\delta}\log\left(\begin{array}{l}\left(\frac{\log\alpha}{\alpha-1}\right)^{\delta}\left(\frac{\alpha\theta^3}{\theta^2+\theta+2}\right)^{\delta}\sum_{r=0}^{\infty}\frac{(-\delta\log\alpha)^r}{r!}\\ \int_0^\infty(1+y+y^2)^\delta e^{-\theta\beta\delta y}\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)^r e^{-\theta ry}dy\end{array}\right) \quad (21)$$

On simplification, we obtain the resulting equation as

$$R(\delta)=\frac{1}{1-\delta}\log\left(\begin{array}{l}\left(\frac{\log\alpha}{\alpha-1}\right)^{\delta}\left(\frac{\alpha\theta^3}{\theta^2+\theta+2}\right)^{\delta}\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\\ \sum_{u=0}^{t}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{r}{s}\binom{s}{t}\binom{t}{u}\binom{\delta}{m}\binom{m}{n}\\ \frac{(-\delta\log\alpha)^r}{r!}\left(\frac{\theta}{\theta^2+\theta+2}\right)^s\\ \left(\frac{2}{\theta}\right)^u\theta^s\int_0^\infty y^{(2s-t+m+n+1)-1}e^{-\theta(r+\delta)y}dy\end{array}\right) \quad (22)$$

After simplifying the expression (22), the Renyi entropy of the distribution is obtained as

$$R(\delta)=\frac{1}{1-\delta}\log\left(\begin{array}{l}\displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{r}{s}\binom{s}{t}\binom{t}{u}\\ \displaystyle\binom{\delta}{m}\binom{m}{n}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\delta}\\ \displaystyle\frac{(-\delta\log\alpha)^{r}}{r\,!}\left(\frac{\theta}{\theta^{2}+\theta+2}\right)^{s+\delta}\left(\frac{2}{\theta}\right)^{u}\\ \displaystyle\theta^{2+s}\,\frac{\Gamma(2s-t+m+n+1)}{(\theta\,(r+\delta))^{2s-t+m+n+1}}\end{array}\right)$$

5.2 Tsallis Entropy

Tsallis entropy [17] of a random variable Y of order λ with pdf (5) is defined by

$$S(\lambda)=\frac{1}{\lambda-1}\left(1-\int_{0}^{\infty}f^{\lambda}(y)dy\right) \tag{23}$$

When equation (5) is substituted into equation (23), the equation (23) becomes

$$S(\lambda)=\frac{1}{\lambda-1}\left(1-\int_{0}^{\infty}\left(\begin{array}{l}\displaystyle\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^{3}}{\theta^{2}+\theta+2}\right)\\ \displaystyle\left(1+y+y^{2}\right)e^{-\theta y}\,\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^{2}+\theta+2}\right)e^{-\theta y}}\end{array}\right)^{\lambda}dy\right)$$

After simplification, we get

$$S(\lambda)=\frac{1}{\lambda-1}\left(1-\left(\begin{array}{l}\displaystyle\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\sum_{m=0}^{\infty}\sum_{n=0}^{m}\binom{r}{s}\binom{s}{t}\binom{t}{u}\binom{\lambda}{m}\\ \displaystyle\binom{m}{n}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\lambda}\left(\frac{\lambda^{r}(-\log\alpha)^{r}}{r\,!}\right)\\ \displaystyle\theta^{2+s}\left(\frac{\theta}{\theta^{2}+\theta+2}\right)^{s+\lambda}\left(\frac{2}{\theta}\right)^{u}\frac{\Gamma(2s-t+m+n+1)}{(\theta\,(r+\lambda))^{2s-t+m+n+1}}\end{array}\right)\right)$$

6. ORDER STATISTICS

The literature on order statistics is very extensive. Order statistics is regarded as one of the most essential tools in non-parametric and inferential statistics. It describes random variables arranged in ascending order of their magnitude. They play important role in many statistical applications including modeling auctions, car races, estimating parameters, optimizing the production process, risk management, reliability of systems, quality control, outlier detection, and many other applied areas. This technique is applicable even for small samples because it is not based on asymptotic theory. The distribution does not compel the distribution to be symmetric, which might be problematic when dealing with extremes where distributions are asymmetric. The highest order statistics is useful in studying floods and meteorological phenomena. Order statistics are more applicable in engineering fields. Life testing provides an ideal illustration of the advantages of order statistics in the case of censored samples.

Consider a set of random variables $Y_1,\ Y_2,\dots,Y_n$ which are independent and identically distributed. The probability density function (pdf) of the random variables provides the distribution of the k^{th} order statistics. The 1^{st} order statistic $Y_{(1)}$ is the set of the minimum values from the set of n random variables. The n^{th} order statistic $Y_{(n)}$ is the set of the maximum values from the set of n random variables.

Suppose $Y_{(1)}, Y_{(2)}, Y_{(3)},\dots,Y_{(n)}$ be the order statistics of a random sample $Y_1, Y_2, Y_3,\dots,Y_n$ obtained from Alpha power transformed Sujatha distribution with pdf $f(y;\theta,\alpha)$ and cdf $F\ (y;\theta,\alpha)$. Then the probability density function of k^{th} order statistic $Y_{(k)}$ can be expressed as

$$f_{Y(k)}(y)=\frac{n!}{(k-1)!\,(n-k)!}\,f_{Y}(y)\left[F_{Y}(y)\right]^{k-1}\left[1-F_{Y}(y)\right]^{n-k} \tag{24}$$

Substituting equations (5) and (6) into equation (24), the pdf of k^{th} order statistic $Y_{(k)}$ of the APS distribution is given by

$$\begin{array}{l}\displaystyle f_{Y(k)}(y)=\frac{n!}{(k-1)!\,(n-k)!}\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^{3}}{\theta^{2}+\theta+2}\right)\left(1+y+y^{2}\right)e^{-\theta y}\\ \displaystyle\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^{2}+\theta+2}\right)e^{-\theta y}}\left(\frac{\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^{2}+\theta+2}\right)e^{-\theta y}}-1}{\alpha-1}\right)^{k-1}\\ \displaystyle\left(1-\frac{\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^{2}+\theta+2}\right)e^{-\theta y}}-1}{\alpha-1}\right)^{n-k}\end{array}$$

The expression for the largest order statistic $Y_{(n)}$ of the Alpha power transformed Sujatha distribution is obtained by putting $k=n$ in equation (24) and is given by

$$f_{Y(n)}(y)=n\left(\begin{array}{l}\displaystyle\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^{3}}{\theta^{2}+\theta+2}\right)\left(1+y+y^{2}\right)\\ \displaystyle e^{-\theta y}\,\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^{2}+\theta+2}\right)e^{-\theta y}}\end{array}\right)\left(\frac{\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^{2}+\theta+2}\right)e^{-\theta y}}-1}{\alpha-1}\right)^{n-1}$$

The expression for the smallest order statistic $Y_{(1)}$ of the

Alpha power transformed Sujatha distribution is obtained by putting $k = 1$ in equation (24) and is given as

$$f_{Y(1)}(y) = n\left(\frac{\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^3}{\theta^2+\theta+2}\right)\left(1+y+y^2\right)e^{-\theta y}}{\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y}}}\right)\left(1-\frac{\alpha^{1-\left(1+\frac{\theta y(\theta y+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y}}-1}{\alpha-1}\right)^{n-1}$$

7. MEASURES OF INEQUALITY

Measure of inequality describes the extent to which a variable under study is distributed unevenly among different proportions of the population. If every proportion of the population receives the same proportion of a variable then such a case is known as the case of perfect equality. Similarly, perfect inequality is defined as a situation wherein a single unit of the population receives the entire amount of the variable under consideration. Although there have been attempts to develop a mathematical tool for measuring the extent of inequality. The first major success was the Lorenz curve due to American economist Lorenz, in the field of economics. Since 1905, the Lorenz curve is a popular tool for analyzing the distribution of income. It is imperative to study inequality because of the fact that disparity in the living standard of the population is the outcome of income inequality. There exist various tools in statistical literature that are used to study the disparity, we discuss some of the well-known and widely used measures of inequality.

7.1 Lorenz and Bonferroni Curves

The Lorenz curve is a graphical representation of the degree of inequality of income in a country. It was an American economist Max O. Lorenz [18] who proposed the Lorenz curve as a method for making the comparison between in income distribution of the population at different time points. The Lorenz curve is represented by a function $L(p)$, where p is the cumulative proportion of income units represented by the horizontal axis. The vertical axis represents the cumulative proportion of the total income received by units, organized from low to high income which is denoted by $L(p)$. The Bonferroni curve [19] is one of the metrics that can be related to the Lorenz curve and has the advantage of being graphically depicted in the unit square (Giorgi and Mondani, [20]). The Bonferroni and Lorenz curves are not just to analyse the relationship between income and poverty in economics. It is also employed in the areas of reliability, medicine, insurance and demography.

The Bonferroni and Lorenz curves for a non-negative random variable Y with density function $f(y)$ is given by equations (25) and (26) as follows

$$B(p) = \frac{1}{p\mu}\int_0^q y\, f(y)\, dy \tag{25}$$

The expression for the Lorenz curve is defined by

$$L(p) = \frac{1}{\mu}\int_0^q y\, f(y)\, dy \tag{26}$$

$$L(p) = p\, B(p)$$

where, $q = F^{-1}(p)$ and $\mu = E(y)$.

The Bonferroni curve of the Alpha power transformed Sujatha distribution is given by

$$B(p) = \frac{1}{\mu_1' p}\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\binom{r}{s}\binom{s}{t}\binom{t}{u}\left(\frac{1}{\theta^2+\theta+2}\right)^{s+1}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{(-\log\alpha)^r}{r!}\right)\left(\frac{2}{\theta}\right)^u\theta^{-t}\left(\frac{1}{r+1}\right)^{2s-t+2}\left(\begin{array}{l}\left(\gamma\left(\theta(r+1)\right)q,(2s-t+2)\right)+\left(\frac{1}{\theta(r+1)}\right)\\ \left(\gamma\left(\theta(r+1)\right)q,(2s-t+3)\right)\\ +\left(\frac{1}{\theta(r+1)}\right)^2\gamma\left(\left(\theta(r+1)\right)q,(2s-t+4)\right)\end{array}\right)$$

The Lorenz curve of the Alpha power transformed Sujatha distribution is given by

$$L(p) = \frac{1}{\mu_1'}\sum_{r=0}^{\infty}\sum_{s=0}^{r}\sum_{t=0}^{s}\sum_{u=0}^{t}\binom{r}{s}\binom{s}{t}\binom{t}{u}\left(\frac{1}{\theta^2+\theta+2}\right)^{s+1}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{(-\log\alpha)^r}{r!}\right)\left(\frac{2}{\theta}\right)^u\theta^{-t}\left(\frac{1}{r+1}\right)^{2s-t+2}\left(\begin{array}{l}\left(\gamma\left(\theta(r+1)\right)q,(2s-t+2)\right)+\left(\frac{1}{\theta(r+1)}\right)\\ \left(\gamma\left(\theta(r+1)\right)q,(2s-t+3)\right)\\ +\left(\frac{1}{\theta(r+1)}\right)^2\gamma\left(\left(\theta(r+1)\right)q,(2s-t+4)\right)\end{array}\right)$$

8. METHODS OF ESTIMATION

This section is devoted to the estimation of parameters based on ordinary least square and maximum likelihood estimation for estimating the unknown parameters of the proposed model. In order to make the inference using parametric estimating techniques, the assumption of the density function must be known. However, in the case of real-life data fitting, the true distribution is not known and prior knowledge is required the approximate the gap between the model and true distribution. Maximum likelihood estimation is one of the most powerful and accepted approaches for making inferences. ML estimate is considered as a consistent estimator that is asymptotically efficient under broad conditions. It was observed to be inefficient in the situation of a mixture of continuous distributions, J-shaped distributions and heavy-tailed distributions (Pitman, [21], Renneby, [22], Cheng, [23]).

8.1 Least Square Estimation: Least square estimate is one of the approaches for estimating the parameters of the model in the case of regression theory. LSE is obtained by minimizing the sum of squared errors between the sample data and the fitted distribution function. Least square estimate of the parameter θ is the estimate $\hat{\theta}$ that minimizes the least square loss function of the form. A disadvantage of the ordinary LSE is that the estimation error is often larger than that of MLE. To reduce the error, weighted LSE is designed.

8.2 Ordinary Least Square Estimation

Swain *et al.* [24] introduced the ordinary least square estimators and weighted least square estimators for the parameter estimation in a beta distribution. The least-square estimators of the distribution $f(.)$ with unknown parameters can be obtained by minimizing (A).

$$S(E(\alpha,\theta)) = \sum_{i=1}^{n}\left(F\left(Y_{(i)}\right)-\left(\frac{i}{n+1}\right)\right)^2 \qquad \text{(A)}$$

where, $F\left(Y_{(i)}\right)$ is the cdf of APS Distribution with $Y_{(i)}$ being the i^{th} order statistic. LSE of the population parameters (θ,α) can be obtained by minimizing the resulting equations,

$$\frac{\partial E(\alpha,\theta)}{\partial\theta} = \sum_{i=1}^{n}\left(\frac{\alpha^{1-\left(1+\frac{\theta y_i(\theta y_i+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y_i}}}{(\alpha-1)}\right)\left(\left(\frac{\alpha^{1-\left(1+\frac{\theta y_i(\theta y_i+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y_i}}-1}{\alpha-1}\right)-\left(\frac{i}{n+1}\right)\right)\log\alpha$$

(27)

$$\frac{\partial E(\alpha,\theta)}{\partial\alpha} = \sum_{i=1}^{n}\left(\frac{\alpha^{-\left(1+\frac{\theta y_i(\theta y_i+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y_i}}}{(\alpha-1)^2}\right)\left(\left(\frac{\alpha^{1-\left(1+\frac{\theta y_i(\theta y_i+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y_i}}-1}{\alpha-1}\right)-\left(\frac{i}{n+1}\right)\right)$$

(28)

Equating equation (27) and (28) to zero and can be computed analytically by using R Software and Mathematica.

8.2 Maximum Likelihood Estimation

To obtain ML estimates of the APS distribution with set of parameters θ and α. Let $y_1, y_2, y_3,\ldots, y_n$ be the observed values from the distribution, then the likelihood function is given by

$$L(y;\theta,\alpha)=\prod_{i=1}^{n}\left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^3}{\theta^2+\theta+2}\right)\left(1+y_i+y_i^2\right)e^{-\theta y_i}\,\alpha^{1-\left(1+\frac{\theta y_i(\theta y_i+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y_i}}\right)$$

The log-likelihood function is obtained as

$$\log L = n\left(\log(\log\alpha)-\log(\alpha-1)+3\log\theta-\log\left(\theta^2+\theta+2\right)\right)+ \sum_{i=1}^{n}\log\left(1+y_i+y_i^2\right)-\theta\sum_{i=1}^{n}y_i+\log\alpha\sum_{i=1}^{n}\left(1-\left(1+\frac{\theta y_i(\theta y_i+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y_i}\right)$$

(29)

The MLE's of θ,α can be obtained by differentiating the equation (29) with respect to the estimators θ and α to get the resulting normal equations.

$$\frac{\partial\log L}{\partial\alpha}=\frac{n}{\alpha\log\alpha}-\frac{n}{\alpha-1}+\frac{1}{\alpha}\sum_{i=1}^{n}\left(1-\left(1+\frac{\theta y_i(\theta y_i+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y_i}\right) \qquad (30)$$

$$\frac{\partial\log L}{\partial\theta}=\frac{3n}{\theta}-\frac{(2\theta+1)n}{\left(\theta^2+\theta+2\right)}-\sum_{i=1}^{n}y_i+\log\alpha\sum_{i=1}^{n}\left(1-\left(1+\frac{\theta y_i(\theta y_i+\theta+2)}{\theta^2+\theta+2}\right)e^{-\theta y_i}\right) \qquad (31)$$

It is quite difficult to solve system of non-linear equations algebraically because of the intricate form of likelihood equations, (30) and (31). The parameters can be estimated using Newton Raphson in R Statistical software [25].

9. APPLICATIONS

In this section, the two real data sets are presented and used to examine the flexibility of Alpha Power Transformed Sujatha (APS) distribution and has been compared with the Two-parameter Sujatha distribution, A new generalized Sujatha (ANGS) distribution, exponential, Amarendra distribution and Sujatha distribution. The goodness of fit statistic is used to know the better model for the fit. The model with lowest AIC, BIC and $AICc$ considers the best model for the said data.

9.1 Data Set I: The data set provides the strength (measured in GPA) of 69 single fiber of 20 mm of gauge length reported by Badar and Priest [26].

9.2 Data Set II: The data set represents the failure times (in hours) of 59 conductors, observed from conducting the accelerated life test and there are not censored observations in the set of items obtained from Lawless [27].

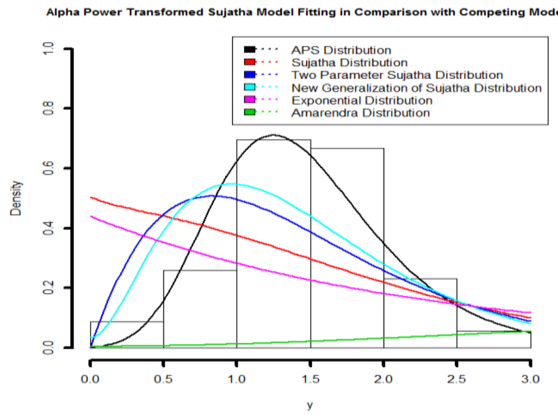
To compare the distributions, we use criteria like AIC (Akaike Information Criterion), $AICc$ (Corrected Akaike Information Criterion) and BIC (Bayesian Information

Criterion). The best distribution would be determined by log L value and $AIC, AICc$ and BIC values.

$$AIC = 2s - 2\log L \quad , \qquad BIC = s\log(n) - 2\log L \quad ,$$
$$AICc = AIC + \frac{2s(s+1)}{n-s-1}$$

where, 's' is the number of estimated parameters, 'n' is the sample size and $-2\log L$ is the maximized value of the log-likelihood function and are shown in table 1.

From table 3, the APS distribution has lower $AIC, BIC, AICc$ values as compared Two-parameter Sujatha distribution, a new generalized Sujatha (ANGSD) distribution, exponential distribution, amarendra distribution and Sujatha distribution. As a result, we conclude that the Alpha power transformed Sujatha distribution (APS) fits better the considered data than the other distributions.



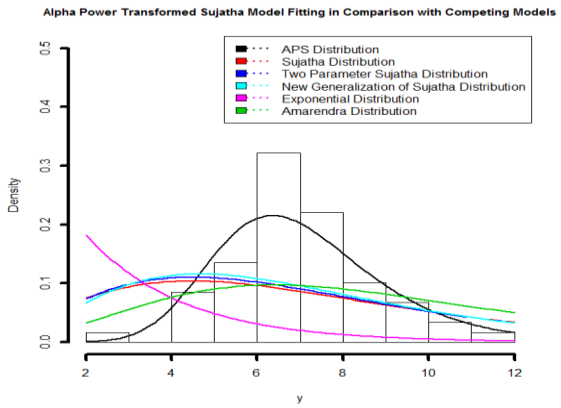


Fig- 5: Graphical data set 1 and 2 fitted by proposed model and related models

10. CONCLUSION

In this study, a new two-parameter distribution has been developed by using the APT technique suggested by Mahdavi and Kundu. The technique adds a shape parameter to the existing model so that it becomes more flexible in data fitting. The proposed distribution is called the Alpha Power Transformed Sujatha distribution. Various structural properties of the distribution have been obtained such as moments, reliability function, hazard function, mean residual time, mean waiting time, order statistics, renyi entropy, Tsallis entropy, Bonferroni and Lorenz curves. The parameters of the distribution are investigated by using the method of maximum likelihood technique. The applicability of the new distribution has been demonstrated with two suitable real-life data sets over one and two-parameter distribution. It has been observed from comparative table 3 that Alpha Power Transformed Sujatha distribution outperforms the other competing distributions for considered data sets in terms of $AIC, BIC, AICc$ and

log- likelihood fitted criteria.

Table-1: Data Set I

0.031	0.314	0.479	0.5520	0.700	0.803	0.861	0.865	0.944	0.958
0.977	1.006	1.021	1.027	1.055	1.063	1.098	1.140	1.179	1.224
1.253	1.270	1.272	1.274	1.301	1.301	1.359	1.478	1.490	1.511
1.382	1.382	1.426	1.434	1.435	1.535	1.554	1.566	1.570	1.586
1.633	1.642	1.648	1.684	1.697	1.726	1.770	1.773	1.800	1.809
1.821	1.848	1.880	1.954	2.012	2.067	2.084	2.090	2.128	2.096
2.233	2.433	2.585	2.585	0.966	1.240	1.514	1.629	1.818	

Table-2: Data Set II

2.997	4.137	4.288	4.531	4.700	4.706	5.009	5.381	5.434	5.807
5.589	5.459	5.640	6.033	6.071	6.087	6.129	6.352	6.369	6.492
6.515	6.522	6.538	6.545	6.573	6.725	6.869	6.923	6.948	6.956
6.958	7.024	7.224	7.365	7.398	7.459	7.489	7.495	7.496	7.543

7.683	7.937	7.945	7.974	8.120	8.336	8.532	8.591	8.687	8.799
9.218	9.254	9.289	9.663	10.092	10.491	11.038			

Table-3: Summary of fitted distributions for data set 1 and 2

Data	Model	ML Estimates		AIC	BIC	AICc	-2lnL
		$\hat{\theta}$	$\hat{\alpha}$				
1	APS	2.7259	5.2008	115.4806	119.9489	115.8499	111.4519
	Two-Parameter Sujatha	1.7499	0.0010	147.2565	151.7247	147.6257	143.2277
	ANGS	2.0428	114.5234	135.6919	140.1601	136.0611	131.6631
	Exponential	0.6911	-	190.9872	193.2213	191.0460	188.9872
	Amarendra	1.7646	-	167.0082	169.2423	167.0670	165.0082
	Sujatha	1.3907	-	170.0589	172.2931	170.4282	168.0446
2	APS	0.8282	7.3327	232.1189	236.2740	232.5552	228.1189
	Two Parameter Sujatha	0.4056	0.0010	292.5018	296.6569	292.9382	288.4718
	ANGS	0.4297	6.1395	286.7689	290.9240	287.2052	282.7689
	Exponential	0.1433	-	349.2809	351.3585	349.3511	347.2809
	Amarendra	0.5332	-	278.3213	280.3988	278.3915	276.3213
	Sujatha	0.3906	-	295.5905	297.6680	296.0268	293.5905

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